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Research Statement

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1 Introduction

Within the broader context of commutative algebra, I study infinite free resolutions of modules over graded rings. Polynomial rings over a field (and quotients thereof) are especially nice because they can be viewed as direct sums of finite dimensional vector spaces, and the same is true for their modules. Not only do such graded rings and modules arise in many settings, including the study of projective varieties, algebraic statistics, and combinatorics, but they also have beautiful properties that make their study interesting in its own right.

In my work, I use free resolutions to detect the structure of a module by iterative approximations. Over a graded ring, every finitely generated module has a presentation as the cokernel of a matrix (the entries of this matrix are the generators of the module). This matrix acts analagously to a linear transformation of vector spaces, and its kernel represents the relations among the generators of the module. The kernel itself is also a module (called a syzygy), and it can be presented in the same way. Repeated, this process produces a sequence of modules called a free resolution. David Hilbert introduced free resolutions in the 1890s to study invariant theory. Today, their uses permeate mathematics, and they are essential to the study of commutative rings and modules.

I also investigate the behavior of free resolutions over different types of graded rings. Are they finite? If not, can they be determined by finite data? If every free resolution over a ring enjoys certain properties, can these properties be leveraged to describe algebraic properties of the ring? We can phrase questions about the numerics of free resolutions by considering Betti tables. A Betti table is a matrix that records the ranks of the free modules in a free resolution and the degrees of their generators. Conjectures and questions about Betti tables can be tantalizingly easy to state. What shape can a Betti table have? What module invariants can be recovered from a Betti table? These questions, and the quest for their answers, are the foundations of my research agenda.

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2 Graded Free Resolutions

Let R be a standard graded $\Bbbk$-algebra. The complex of graded free modules

$$0 \longleftarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{b_{0,j}} \xleftarrow{\partial_0} \bigoplus_{j\in\mathbb{Z}} R(-j)^{b_{1,j}} \xleftarrow{\partial_1} \bigoplus_{j\in\mathbb{Z}} R(-j)^{b_{2,j}} \xleftarrow{\partial_2} \cdots$$

is called a minimal graded free resolution of M provided it satisfies the following three conditions: (i) ∂_i is a matrix of homogeneous non-units, (ii) $\operatorname{im}\partial_{i+1} = \ker\partial_i$, and (iii) $M = \operatorname{coker}\partial_0$. Each $b_{i,j}$ is independent of choice of minimal free resolution, and we call these numbers collectively the ***graded Betti numbers of*** M. The ***Betti table of*** M is a matrix

$$\beta(M) = \begin{pmatrix} \vdots & \vdots & \vdots & \\ b_{0,0} & b_{1,1} & b_{2,2} & \cdots \\ b_{0,1} & b_{1,2} & b_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

whose entry in the i-th column and j-th row is $b_{i,j+i}$.

In general, it is not possible to determine exactly when a matrix of integers is the Betti table of a module. But what if we're allowed to scale Betti tables? In 2006, M. Boij and J. Söderberg [9] had the audacious idea to consider Betti tables "up to rational multiple." This insight developed into a program of embedding Betti tables into a rational vector space and analyzing the convex cone that they span. If for a ring R we find a generating set for the cone, then we are able to answer the question, "When does a ray contain the multiple of a Betti table?" (See, for example, [10, 15, 16, 19] for the foundational papers in this field, and [24] for an expository survey.)

We define ***the cone of Betti tables over*** R by

$$\mathrm{B}_{\mathbb{Q}}(R) = \left\{ \sum_{\text{finite}} q\beta(M) \,\middle|\, M \text{ is a finitely generated } R\text{-module, } q \in \mathbb{Q}_{\geq 0} \right\}.$$

Hypersurfaces. A ***hypersurface*** is a ring of the form $\Bbbk[x_1, \ldots, x_n]/(f)$ where f is a homogeneous polynomial. Given a hypersurface, many modules will have infinite resolutions, which makes studying Betti tables inherently more challenging than over polynomial rings. However, after initial unruliness, these resolutions are eventually periodic after a fixed number of steps [14]. Therefore every Betti table can be described with a finite amount of data, similar to Betti tables over a polynomial ring.

With this in mind, C. Berkesch, J. Burke, D. Erman, and I began thinking about extending Boij-Söderberg theory to hypersurfaces. We provided the first extension of Boij-Söderberg theory to a ring that is not a polynomial ring.

Theorem 1 (Berkesch-Burke-Erman-Gibbons, [6])**.** *Let q be any quadratic form in $\Bbbk[x, y]$, and let $R = \Bbbk[x, y]/(q)$. The Betti table of each R-module can be written as a nonnegative rational combination of:*

1. *the Betti tables of those finitely generated modules having finite $\Bbbk$-dimension and a minimal free resolution of the form*
$$0 \longleftarrow R(-d_0) \longleftarrow R(-d_1) \longleftarrow 0;$$
2. *the Betti tables of those finitely generated modules having finite $\Bbbk$-dimension and a minimal free resolution of the form*
$$0 \longleftarrow R(-d_0) \longleftarrow R(-d_1) \longleftarrow R(-d_1 - 1) \longleftarrow R(-d_1 - 2) \longleftarrow \cdots.$$

Moreover, the set of modules above is minimal: none can be written as a rational combination of the others.

The hypersurface $\Bbbk[x, y]/(q)$ has one feature that makes it more attractive to work with than a hypersurface $\Bbbk[x, y]/(f)$ with $\deg(f) > 2$: it is a ***Koszul algebra***, which means that over R, the nonzero graded Betti numbers of $\Bbbk$ occupy exactly one row of the Betti table. Over a graded ring R, a nonzero R-module M is said to be ***linear*** provided $b_{i,j}(M) = 0$ for all $j - i \neq d$. A linear module generated in degree 0 is said to be a ***Koszul module***.

Short Gorenstein Rings. A (graded) $\Bbbk$-algebra R with maximal ideal $\mathfrak{m}$ is called ***short*** provided $\mathfrak{m}^3 = 0$; when R is Gorenstein, this is equivalent to the condition that $H_R(s) = 1 + es + s^2$ where $e = \dim_{\Bbbk}(R_1)$. The integer e is called the ***multiplicity of*** R. Short Gorenstein $\Bbbk$-algebras are Koszul [13], which is helpful for overcoming some of the difficulties inherent to detecting indecomposable modules.

In [3], L. Avramov, S. Iyengar, and L. Şega show that over short Gorenstein rings, the nonlinear indecomposable modules are classified by their Hilbert series and by their Betti tables. These modules have some structure at the level of Betti tables that make them easy to detect; they have nonzero Betti numbers in one row for a finite number of columns, and then the nonzero Betti numbers "jump" to the next row and stay there (forever; these modules have infinite free resolutions). Linear modules, on the other hand, are exactly those modules that have nonzero Betti numbers that never wander from a single row. To handle the linear modules, we use Matlis duality. Take E to be the (graded) injective hull of $\Bbbk$: the R-module E looks just like R but has "upside down" grading because we identify $\Bbbk$ and the socle of R. For a short Gorenstein graded ring, $E_i = R_{2-i}$. Take $(-)^* = \operatorname{Hom}_R(-, E)$.

Theorem 2 (Avramov-Gibbons-Wiegand, [2])**.** *Let R be a short Gorenstein $\Bbbk$-algebra with multiplicity $e \geq 2$, and let M be an indecomposable R-module. Define the sequence (b_n) via $b_n = b_{n,n}(\Bbbk)$ and set $b_{-1} = 0$. Fix an integer $0 \leq q \leq e - 1$. The following statements are equivalent:*

(i) $M = \operatorname{Syz}_{-n-1}((R/I)^*)$ *for an ideal I minimally generated by $e - q$ linear forms.*

(ii) $H_M(s) = b_n - qb_{n-1} + (b_{n+1} - qb_n)s$

(iii) *M is nonlinear if and only if $q = 0$.*

This theorem leads to an algorithm for finding the nonlinear indecomposable summands a module M over a short Gorenstein ring from a finite initial window of $\beta(M)$. We are also able to determine some invariants of the remaining linear summands. These results come from understanding the structures of the semigroup of Betti tables and the rational cone of Betti tables in [2].

The allure of any algorithm is proportionate to how easy it is to encode; with Branden Stone, I have been working on extending the Macaulay2 "Boij-Soederberg" package to include alternative decomposition algorithms. Our work for [20] can be followed on GitHub.

Future Work. In future work, my goal is to find similar results for other Koszul algebras. Results about modules over a Koszul algebra can sometimes yield information about the representation theory for the Koszul dual [3]. For example, a Koszul module M with Hilbert series $p + qs$ over a short ring R induces a module $\mathcal{M}$ over the Koszul dual $\mathcal{E}$ with presentation

$$0 \to \mathcal{E}(1)^q \to \mathcal{E}^p \to \mathcal{M} \to 0.$$

I would like to better understand what indecomposability of a module M over R implies for the module $\mathcal{M}$ over $\mathcal{E}$. The ring $\mathcal{E}$ need not be commutative; with this in mind, Andrew Conner, W. Frank Moore and I are writing a noncommutative algebra package for [20].

3 Graph Theoretic Results

Of recent interest to me is the intersection of graph theory and commutative algebra, although my prior results are more in the mainstream of graph theory.

One way to learn about graph's structure is to study its automorphisms. A ***graph automorphism*** of a graph G is an edge-preserving permutation of the vertices of G. In [25], Josh Laison and I describe a way to study graph automorphisms is by systematically "pinning down" certain vertices. Intuitively, the fixing number of G is the minimum number of vertices one must pin down to eliminate nontrivial automorphisms. The ***fixing set*** of a (finite) group Γ is

$$\operatorname{fix}(\Gamma) = \{\operatorname{fix}(G) \,|\, G \text{ is a finite graph with } \operatorname{Aut}(G) \cong \Gamma\}.$$

In our paper, we classify the fixing sets of abelian groups, find bounds on the fixing sets of symmetric groups, examine several special graphs, and pose conjectures for future work. We also show that our notion of a fixing set for a graph is equivalent to Debra Boutin's definition of a ***determining set***, which appeared in the literature at about the same time [11].

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