RECURSIVE STRATEGY FOR DECOMPOSING BETTI DIAGRAMS OF COMPLETE INTERSECTIONS

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Let $S = \mathbb{k}[x_1, x_2, \ldots, x_d]$ (standard graded $\mathbb{k}$-algebra over a field $\mathbb{k}$).

Let M be a finitely generated graded S-module with minimal graded free resolution

$$F_{.} : 0 \leftarrow \oplus_j S(-j)^{\beta_{0,j}(M)} \leftarrow \oplus_j S(-j)^{\beta_{1,j}(M)} \leftarrow \cdots \leftarrow \oplus_j S(-j)^{\beta_{d,j}(M)} \leftarrow 0,$$

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The **Betti diagram** of an S-module M tabulates the ranks of the free modules in the free resolution of M:

$$\beta(M) = \begin{array}{c|cccccc} & 0 & 1 & \cdots & i & \cdots & n \\ \hline 0 & \beta_{0,0}(M) & \beta_{1,1}(M) & \cdots & \beta_{i,i}(M) & \cdots & \beta_{n,n}(M) \\ 1 & \beta_{0,1}(M) & \beta_{1,2}(M) & \cdots & \beta_{i,i+1}(M) & \cdots & \beta_{n,n+1}(M) \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ j & \beta_{0,j}(M) & \beta_{1,1+j}(M) & \cdots & \beta_{i,i+j}(M) & \cdots & \beta_{j,n+j}(M) \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \end{array}.$$

An S-module M is called **pure** if there is a **degree sequence** $\mathbf{d} = (d_0 < d_1 < \cdots < d_n)$ such that $\beta_{i,j}(M) = 0$ if $j \neq d_i$.

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Notation: $\pi(\mathbf{d})$ is the Betti diagram of the* pure module M with the associated degree sequence $\mathbf{d} = (d_0 < d_1 < \cdots < d_n)$ with a (technical) scaling factor. Given two degree sequences $\mathbf{c}$ and $\mathbf{d}$, we say $\mathbf{c} \leq \mathbf{d}$ if $c_i \leq d_i$ for each i.

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Theorem (Boij-Söderberg ($n \leq 2$), Eisenbud-Schreyer (all n))

For every S-module M, there exists a unique list of totally ordered degree sequences $\mathbf{d}^1 < \cdots < \mathbf{d}^r$ so that

$$\beta(M) = \sum q_i \pi\left(\mathbf{d}^i\right)$$

where $q_i \in \mathbb{Q}_{\geq 0}$.

Example ($M = S/(x, y^2, z^2)$)

$\beta(M) =$

	0	1	2	3
0:	1	1	.	.
1:	.	2	2	.
2:	.	.	1	1

Example ($M = S/(x, y^2, z^2)$)

$$\beta(M) = \begin{array}{r|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & 1 & 1 & . & . \\ 1: & . & 2 & 2 & . \\ 2: & . & . & 1 & 1 \end{array}$$

$$= 8 \begin{array}{r|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & \frac{1}{15} & \frac{1}{8} & . & . \\ 1: & . & . & \frac{1}{12} & . \\ 2: & . & . & . & \frac{1}{40} \end{array} +8 \begin{array}{r|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & \frac{1}{30} & . & . & . \\ 1: & . & \frac{1}{6} & \frac{1}{6} & . \\ 2: & . & . & . & \frac{1}{30} \end{array} +8 \begin{array}{r|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & \frac{1}{40} & . & . & . \\ 1: & . & \frac{1}{12} & . & . \\ 2: & . & . & \frac{1}{8} & \frac{1}{15} \end{array}$$

$$= 8 \cdot \pi(0, 1, 3, 5) \quad +8 \cdot \pi(0, 2, 3, 5) \quad +8 \cdot \pi(0, 2, 4, 5).$$

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Elimination Table:

	0	1	2	3
0:	3	1	.	.
1:	.	3	2	.
2:	.	.	3	3

The Betti diagram of a complete intersection $M = S/(f_1, \dots, f_d)$ over the ring S is determined by the degrees of its minimal generators.

Example

If $S = \Bbbk[x]$ and $M = S/(f)$, then $\beta(M) = \deg(f) \cdot \pi\,(0, \deg(f))$.

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If $S = \Bbbk[x, y]$ and $M = S/(f, g)$, then

$$\begin{aligned}\beta(M) = {} & \deg(f)\deg(g) \cdot \pi\,(0, \deg(f), \deg(f) + \deg(g)) \\ & + \deg(f)\deg(g) \cdot \pi\,(0, \deg(g), \deg(f) + \deg(g))\,.\end{aligned}$$

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Question

For $S = \Bbbk[x_1, \dots, x_d]$ and any complete intersection M where

$$\beta(M) = q_1\pi\left(\mathbf{d}^{(1)}\right) + \cdots + q_r\pi\left(\mathbf{d}^{(r)}\right),$$

is there a uniform formula for determining q_j and $\mathbf{d}^{(j)}$ in terms of $\deg(f_i)$?

Proposition (GJMRSW 2015)

Let S be $\Bbbk[x_1, x_2, x_3]$, and let $I = (f_1, f_2, f_3)$ be an ideal generated by a homogeneous regular sequence with $\deg(f_i) = a_i$ where $a_i \leq a_{i+1}$ for all i. Then

$$\begin{aligned}\beta(S/I) = \quad & a_1a_2(a_2+a_3)\cdot\pi\left(0, a_1, a_1+a_2, a_1+a_2+a_3\right)\\ &+ a_1a_2(a_3-a_1)\cdot\pi\left(0, a_2, a_1+a_2, a_1+a_2+a_3\right)\\ &+ 2a_1a_2(a_1+a_3-a_2)\cdot\pi\left(0, a_2, a_1+a_3, a_1+a_2+a_3\right)\\ &+ a_1a_2(a_3-a_1)\cdot\pi\left(0, a_3, a_1+a_3, a_1+a_2+a_3\right)\\ &+ a_1a_2(a_2+a_3)\cdot\pi\left(0, a_3, a_2+a_3, a_1+a_2+a_3\right).\end{aligned}$$

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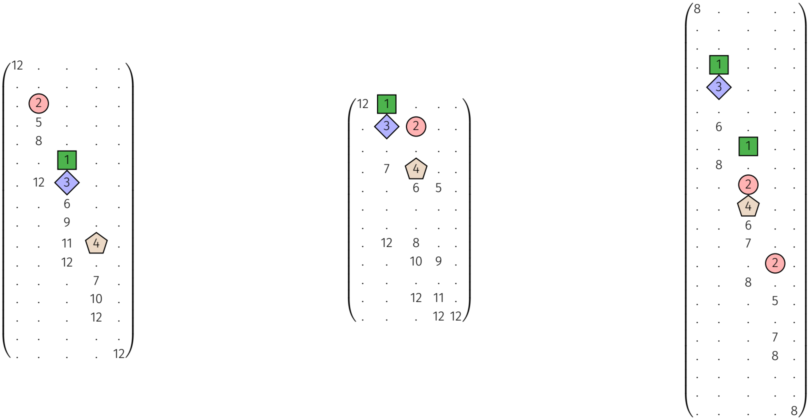
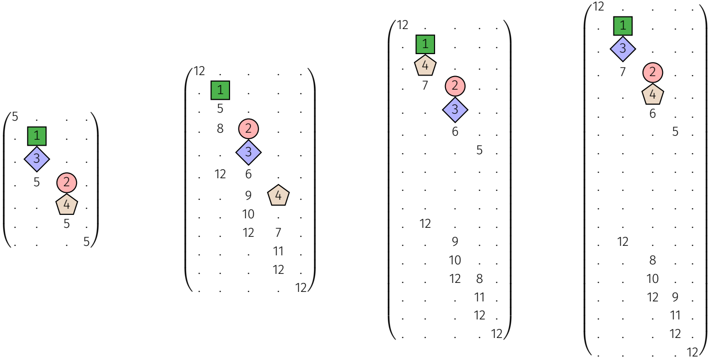


$I = (x^3, y^4, u^5, v^7)$ $\quad$ $J = (x, y^2, u^4, v^8)$ $\quad$ $K = (x^4, y^5, u^7, v^9)$



$I = (x^2, y^3, u^4)$ $\quad$ $I_6 = I + (v^6)$ $\quad$ $I_{11} = I + (v^{11})$ $\quad$ $I_{13} = I + (v^{13})$

Let $c > 1$. Consider a complete intersection

$$R = \Bbbk[x_1, \ldots, x_c, x_{c+1}]/(x_1^{a_1}, \ldots, x_c^{a_c}, x_{c+1}^{a_{c+1}}).$$

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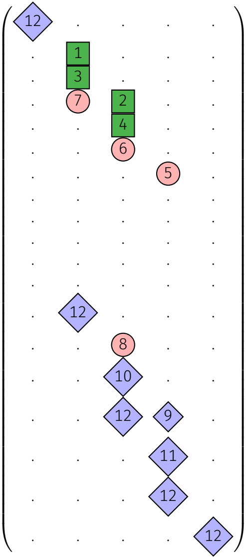
EXAMPLE



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main theorem (ghs 2017)

Assume the betti diagram of the complete intersection generated in degrees $a_1 \leq \ldots \leq a_c$ with BS-decomposition

$$\beta(a_1, \ldots, a_c) = \sum_{s=1}^{\varepsilon} z_s \pi(\mathbf{d}^s)$$

has no instances of mass elimination. If

$$D = \beta(a_1, \ldots, a_c, a_{c+1}),$$

then for a_{c+1} large enough, the above algorithm produces the BS decomposition of D and the coefficients determined by Phase 1 and Phase 3 are linear functions of the z_i's.

Consider $\beta = \beta(a_1, \ldots, a_c, a_{c+1})$ with a_{c+1} large enough. Then

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2. the elimination order of β stabilizes;
3. the coefficients from Phase 1 and Phase 3 are given by linear polynomials in a_{c+1};
4. the recursive algorithm produces the original BS decomposition;
5. the number of terms in the BS decomposition of β is constant.

DETAILS

If $a = \sum_{i=1}^{c} a_i$ and $a_{c+1} > \max\{a, \frac{r_i}{z_i}, \ldots, \frac{r_\varepsilon}{z_\varepsilon}\}$, then

$$
\begin{aligned}
\beta = &\sum_{s=1}^{\varepsilon-1}(z_s a_{c+1} - r_s)\pi(\mathbf{e}^s) \\
&+ \sum_{s=\varepsilon+1}^{\delta} (c!)(a_1 \ldots a_c)\left(a_{c+1} - \left(\sum_{i=1}^{s-n} a_{c+1-i} - a_i\right)\right)\pi(\mathbf{e}^s) \\
&+ \sum_{s=1}^{\varepsilon-1}(z_{\varepsilon-s+1} a_{c+1} - r_{\varepsilon-s+1})\pi(\mathbf{e}^{\varepsilon-s+1})^*
\end{aligned}
$$

where

$$
r_k = \left(\frac{j_k - a}{p_k} b_k - \sum_{s=1}^{k-1} \frac{\pi(\mathbf{d}^s)_{i_k, j_k}}{p_k} r_s\right) \text{ and } \beta(a_1, \ldots, a_c) = \sum_{s=1}^{\varepsilon} z_s \pi(\mathbf{d}^s).
$$

Thanks!