Convex Geometric Graphs with No Short Self-intersecting Paths

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Abstract

Pach, Pinchasi, Tardos and Tóth proved that in a straight-line graph drawing in which no path of length 3 crosses itself (called locally planar) the number of edges can be superlinear in the number of vertices. In contrast, this paper shows that if the vertices form a convex set such a graph drawing (here named locally outerplanar) has at most a linear number of edges. As an important development toward the proof, this paper also shows that every locally outerplanar graph has a vertex of degree at most 2.

1 Introduction

A straight-line graph drawn in the plane with vertices in general position is called a geometric graph. Much of the work in geometric graph theory is extremal in nature. The canonical question is: "What is the maximum number of edges that a geometric graph on n vertices can have without containing a given geometric subgraph?" Some of the forbidden subgraphs that have been studied are: sets of pairwise disjoint edges and sets of pairwise crossing edges, non-crossing cycles and self-intersecting cycles, non-crossing paths and self-intersecting paths. A survey of these and other results in geometric graph theory is provided in [1].

The forbidden subgraphs we focus on here are self-intersecting paths. Pach, Pinchasi, Tardos and Tóth [2] proved that geometric graphs on n vertices with no self-intersecting paths of length 3 have at most $O(n \log n)$ edges and that this bound is asymptotically tight. Tardos [3] has a method for constructing geometric graphs with no self-intersecting paths of length $2k+1$ and $\theta(n \log^{(k)} n)$ edges, where $\log^{(k)}$ is the k-times-iterated log function. The superlinear number of edges means that none of these graph classes is d-degenerate for any fixed d.

A geometric graph with no short self-intersecting path is sometimes called "locally planar." In a similar vein, this paper defines a "locally outerplanar graph" - a geometric graph on a convex set of vertices with no short self-intersecting path. The results on locally outerplanar graphs provide interesting contrast to those for locally planar graphs. Theorem 1 tells us that locally outerplanar graphs are 2-degenerate; they are guaranteed to have a vertex of degree 2 or less. An immediately corollary tells us that locally outerplanar graphs are 3-colorable, have vertex and edge arboricity 2, and have at most $2n-3$ edges where n is the number of vertices. Notice that this bound is the same as that for outerplanar graphs. Theorem 2 tells us that a locally outerplanar graph with at least one crossing has strictly fewer edges than a maximal outerplanar graph on the same number of vertices; it has no more than $2n-6$ edges.

This paper is organized as follows: Section 2 provides the definition of local outerplanarity, makes some basic observations, and sets out notation and terminology that is useful in the remainder of the paper. In Section 3 we see the proof that a locally outerplanar graph has a vertex of degree at most 2. Section 4 provides the proof that a locally outerplanar graph with n vertices and at least one crossing has at most $2n-6$ edges.

2 Local Outerplanarity

A *geometric graph* is a straight-line graph drawn in the plane so that no three vertices lie on a single line and no three edges intersect at a single point. A *convex geometric graph* is a geometric graph all of whose vertices lie on the boundary of its convex hull. We use the notation $\overline{G}$ to denote a geometric graph and reserve the notation G for the underlying abstract graph. A geometric graph can be called k-locally planar if it has no self-intersecting path of length k. This leads us to define a k-locally outerplanar graph as a convex geometric graph with no self-intersecting path of length k. As k grows the graph drawings become increasingly planar-like or outerplanar-like; we expect larger k to provide stronger results. However the results in this paper are sufficiently strong that larger k would not provide improvement. So for this paper we call a geometric graph *locally planar*, and a convex geometric graph *locally outerplanar*, if they contain no self-intersecting paths of length 3.

Example 1. On the left in Figure 1 is a locally outerplanar graph; it has self-intersecting paths of length 4 but no self-intersecting path of length 3. On the right is a convex geometric graph that is not locally outerplanar; it does contain self-intersecting paths of length 3.

Notice that in both cases the underlying abstract graph is outerplanar but that the given graph drawing has a crossing. In geometric graph theory

Figure 1:

we cannot assume that an outerplanar graph is drawn without crossings. When it is, we emphasize this by calling it an *outerplanar graph drawing.*

Observation 1. A locally outerplanar graph whose underlying abstract graph is not outerplanar has seven or more vertices.

If G is not outerplanar it must contain a subdivision of $K_{2,3}$ or of K_4 [4]. Each of K_4 and $K_{2,3}$ can be drawn as a convex geometric graph with just one crossing. This convex drawing of $K_{2,3}$ requires two extra vertices to turn the self-intersecting paths of length 3 into paths of length 4. The convex drawing of K_4 requires 4 extra vertices to obtain a locally outerplanar graph. These are illustrated in Figure 2.

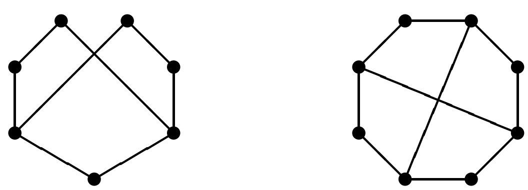

Figure 2:

The following notation, terminology, and observations will be useful in the remainder of of the paper.

Notation, Terminology

Let $\overline{G}$ be a connected convex geometric graph on n vertices. Label the vertices $1, \cdots, n$ moving clockwise around the outside of the convex hull.

A path P of length m is given by a set of $m + 1$ distinct vertices $a_0 a_1 \cdots a_m$ where each $a_j a_{j+1}$ is an edge of $\overline{G}$. It is sometimes useful to denote P by $a_0 P' a_m$ where P' is the subpath $a_1 \cdots a_{m-1}$.

Observation 2. Let ij and $k\ell$ be distinct edges of $\overline{G}$. We may assume that $i < j$ and $k < \ell$ and $i \leq k$. Because the vertices of $\overline{G}$ are labeled cyclically, edges ij and $k\ell$ cross if and only if $i < k < j < \ell$. Similarly if $a_1 \cdots a_r$ and $b_1 \cdots b_s$ are vertex-disjoint paths of $\overline{G}$ where $a_1 < a_r$ and $b_1 < b_s$ and $a_1 < b_1$, then these paths cross if $b_1 < a_r < b_s$.

We say $P = a_0 \cdots a_m$ is a *simple self-intersecting path* if its first and last edges cross and these are the only edges of P to cross. That is, a_0a_1 and $a_{m-1}a_m$ provide the only crossing in P.

Observation 3. If $P = a_0P'a_m$ is a simple self-intersecting path then the vertices of P' are labeled cyclically. That is, if a_0 is larger than all other a_i then either $a_1 < \cdots < a_{m-1}$ or $a_{m-1} < \cdots < a_1$. We can assume that P is written so that P' is traversed clockwise.

To see this, assume that $a_0 = \max\{a_0, a_1, a_{m-1}, a_m\}$ (or rotate the vertex labels so that this is true). For the moment assume that $a_m < a_{m-1}$. By Observation 2, since a_1a_0 crosses a_ma_{m-1}, we have $a_m < a_1 < a_{m-1}$. In particular, a_1 is smaller than a_{m-1}. We want to show that the rest of the a_j in P' fall, in order, between a_1 and a_{m-1}. Again by Observation 2, if there exits j between 2 and $m-2$ (inclusive) so that a_j is smaller than a_1 then the path $a_j \cdots a_{m-1}$ crosses a_1a_0 – a contradiction. Similarly if a_j is larger than a_{m-1} then a_ma_{m-1} crosses $a_1 \cdots a_j$. So when j is between 2 and $m-2$, a_j lives clockwise between a_1 and a_{m-1}. Suppose two of the internal vertices of P' are out of order. If there exist $j < k$ between 2 and $m-2$ so that $a_j > a_k$, then the paths $a_1 \cdots a_j$ and $a_k \cdots a_{m-1}$ cross. Thus we have $a_1 < \cdots < a_{m-1}$ as desired. Similarly, if $a_{m-1} < a_m$ we can show that $a_{m-1} < \cdots < a_1$. We can choose to write P so that P' is traversed clockwise. Unless stated otherwise we always assume this is the case.

Call an ordered pair of vertices (u, v) a *corner pair* if there exists a simple self-intersecting path $P = a_0P'a_m$ so that u is the initial point of P', v is the terminal point of P', and P' is traversed clockwise. (See Figure 3, path $P = hbcfga$, for an example.) Since we are working with locally outerplanar graphs there is outerplanar behavior "local" to P' – it is this behavior we wish to capture. Consider the the subgraph induced by the vertices located clockwise between u and v inclusive. P' is contained in one connected component of this subgraph; denote this component by $\overline{H}_{u,v}$. Note that $\overline{H}_{u,v}$ depends only on the corner pair (u, v), not on the path P; $\overline{H}_{u,v}$ contains every non-crossing subpath of every simple self-intersecting path with corner pair (u, v).

Call the corner pair (u, v) *minimal* if it is the only corner pair within $V(\overline{H}_{u,v})$. In this situation we call u and v the *corner vertices* of $\overline{H}_{u,v}$ and all other vertices of $\overline{H}_{u,v}$ we call *non-corner vertices.* If (u, v) is a minimal corner pair, there is no edge from a non-corner vertex of $\overline{H}_{u,v}$ to a vertex of $\overline{G}$ located counterclockwise between u and v (exclusive); such an edge would necessarily cross an edge incident with u or v, providing an additional corner pair. Note that by definition if (u, v) is not minimal, there is a minimal corner pair within $V(\overline{H}_{u,v})$. Call a simple self-intersecting path a *minimal self-intersecting path* if its corner pair is minimal. Notice that there may be many minimal self-intersecting paths with the same minimal corner pair.

Example 2. Consider the partial convex geometric graph in Figure 3. Notice that (b, g) is a minimal corner pair and $P = hbcfga$ is a minimal self-intersecting path. $\overline{H}_{b,g}$ is the subgraph induced by $\{b, c, d, f, g\}$. The corner vertices of $\overline{H}_{b,g}$ are b, g, and the non-corner vertices are c, d, f. Notice that the vertex e is not in $\overline{H}_{b,g}$ because it is not "local" to the paths we are considering. Further notice that $\overline{H}_{b,g}$ is an outerplanar graph drawing. We see later that this is always true when the corner pair is minimal.

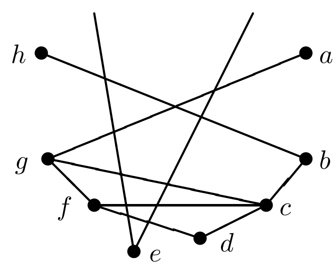

Figure 3:

3 Degeneracy

In contrast to locally planar graphs (which are not d-degenerate for any fixed d) the following theorem says that locally outerplanar graphs are 2-degenerate (or 3-degenerate, depending on your preferred method of counting degeneracy).

Theorem 1. *If $\overline{G}$ is a locally outerplanar graph then it has a vertex of degree at most* 2.

Proof. Let $\overline{G}$ be a convex geometric graph with no self-intersecting path of length 3. We may assume that $\overline{G}$ is connected. Label the vertices $1, \cdots, n$ moving clockwise around the outside of the convex hull.

If there is no self-intersecting path in $\overline{G}$ then G is outerplanar. It is well-known that every outerplanar graph has two non-adjacent vertices of degree at most 2, giving us our result. Thus we may assume that there is a self-intersecting path in $\overline{G}$. Choose a minimal self-intersecting path $P = a_0 \cdots a_m = a_0 P' a_m$. We see later that the minimality condition ensures that $\overline{H}_{a_1,a_{m-1}}$ is an outerplanar graph drawing.

Recall that the subgraph $\overline{H}_{a_1,a_{m-1}}$ is the connected component containing P' of the subgraph induced by the vertices of $\overline{G}$ that are located clockwise between a_1 and a_{m-1} inclusive. It is within $\overline{H}_{a_1,a_{m-1}}$ that we will find our vertex of small degree. For simplicity of notation, denote $\overline{H}_{a_1,a_{m-1}}$ by $\overline{H}$. Also recall that non-corner vertices of $\overline{H}$ have no neighbors located counterclockwise between a_1 and a_{m-1} (exclusive). Thus in $\overline{G}$ all neighbors of non-corner vertices of $\overline{H}$ fall clockwise between a_1 and a_{m-1} and therefore are in $\overline{H}$. In particular, the non-corner vertices of $\overline{H}$ have the same degree in $\overline{H}$ that they have in $\overline{G}$.

Now we wish to prove that $\overline{H}$ is an outerplanar graph drawing. To do this we will show that if $\overline{H}$ contains a self-intersecting path, then it contains a corner pair other than (a_1, a_{m-1}) – a contradiction of minimality. If $\overline{H}$ contains a crossing, it contains a simple self-intersecting path. By our minimality assumption, the corner pair must be (a_1, a_{m-1}). We may denote an associated pair of crossing edges by $a_1 x$ and $y a_{m-1}$ where $a_1 < y < x < a_{m-1}$. If x is not on P' there exists i so that $a_i < x < a_{i+1}$ and therefore $a_1 x$ crosses $a_i a_{i+1}$. Then the path $x a_1 a_2 \cdots a_i a_{i+1}$ is a self-intersecting path with corner pair (a_1, a_i) – a contradiction. We get the same type of contradiction if y is on P'. Thus we may assume that both $x = a_i$ and $y = a_j$ are on our subpath P'. But then $a_1 a_i a_{i-1} \cdots a_{j+1} a_j a_{m-1}$ is a self-intersecting path with corner pair (a_i, a_j) – again a contradiction. Thus $\overline{H}$ contains no crossing and is an outerplanar graph drawing.

Since $\overline{G}$ contains no self-intersecting path of length 3, the crossing of edges $a_0 a_1$ and $a_{m-1} a_m$ guarantees that $a_1 a_{m-1}$ is not an edge in $\overline{G}$. Let $\overline{H}'$ be the convex geometric graph obtained by adding edge $a_1 a_{m-1}$ to $\overline{H}$. All vertices of $\overline{H}$ fall clockwise between a_1 and a_{m-1} so the new edge crosses no edge of $\overline{H}$. Thus the abstract graph underlying $\overline{H}'$ is outerplanar and as such has two non-adjacent vertices of degree less than or equal to 2. Since a_1 and a_{m-1} are adjacent in $\overline{H}'$ this means at least one of the vertices of small degree must be in the non-empty set of non-corner vertices of $\overline{H}$. Thus one of the non-corner vertices of $\overline{H}$ has degree at most 2 in $\overline{H}$ and therefore in $\overline{G}$. $\square$

4 Number of Edges

Theorem 1 can be combined with straightforward induction proofs to yield quick results about locally outerplanar graphs. These results are identical to those for outerplanar graphs.

Corollary 1.1. *A locally outerplanar graph on n vertices has at most $2n-3$ edges, is 3-colorable, and has both vertex and edge arboricity 2.*

The fact that a locally outerplanar graph has no more edges than a maximal outerplanar graph is surprising. The set of graphs that have locally outerplanar drawings is strictly larger than the set of outerplanar graphs – so we expect to be able to use more edges. But not only do locally outerplanar graphs not have more edges than maximal outerplanar graphs, the even the densest has *fewer*. In particular, in Theorem 2 we prove that a locally outerplanar graph with n vertices and at least one crossing has at most $2n-6$ edges – and this bound is sharp. To make the induction proof for Theorem 2 short and simple we need the following lemma.

Lemma 1. *Suppose $\overline{G}$ is a locally outerplanar graph with n vertices and at least one crossing. If $\overline{G}$ contains a vertex x of degree at most 2 so that $\overline{G} \setminus \{x\}$ is an outerplanar graph drawing, then $\overline{G}$ has at most $2n-6$ edges.*

Proof. Let $\overline{G}'$ denote $\overline{G} \setminus \{x\}$, an outerplanar graph drawing. Then there is a (not necessarily unique) set of edges whose addition to $\overline{G}'$ yields a maximal outerplanar graph drawing. Let R be such an edge set. Color the edges of $\overline{G}$ blue, the edges of R red, and let $\overline{M} = \overline{G} \cup R$. Then $\overline{M} \setminus \{x\} = \overline{G}' \cup R$ is a 2-edge colored maximal outerplanar graph drawing which we denote $\overline{M}'$.

We will carefully study the edges of $\overline{M}'$ that cross edges incident with x. These crossings indicate self-intersecting paths of length 3 in $\overline{M}$. Since $\overline{G}$ is locally outerplanar, none of these can be monochromatically blue. We use this fact repeatedly to determine the minimum number of edges in R.

Example 3. Figure 4 shows such a graph. The edges incident with x are dashed. The edges of R are dotted. The edges of $\overline{G}'$ are solid. The maximal outerplanar graph drawing $\overline{M}'$ is the union of the solid and the dotted lines.

In summary we have:

- $\overline{G}$ is a locally outerplanar graph with n vertices, at least one crossing, and a vertex x of degree at most 2 whose removal yields an outerplanar graph drawing.
- $\overline{G}' = \overline{G} \setminus \{x\}$ is an outerplanar graph drawing with $n-1$ vertices.

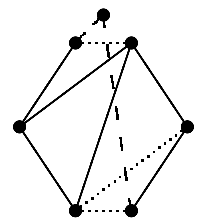

Figure 4:

- R is a set of edges whose addition to $\overline{G}'$ yields a maximal outerplanar graph drawing.
- $\overline{M} = \overline{G} \cup R$ is a convex geometric graph with n vertices and $|E(\overline{G})| + |R|$ edges.
- $\overline{M}' = \overline{M} \setminus \{x\} = \overline{G}' \cup R$ is a 2-edge colored maximal outerplanar graph drawing with $n-1$ vertices, $|E(\overline{G}')|$ blue edges, and $|R|$ red edges.

Since $\overline{M}'$ is maximal outerplanar on $n-1$ vertices, it has $2(n-1)-3 = 2n-5$ edges. Then $\overline{G}$ has $2n-5+\deg(x)-|R|$ edges. If either $\deg(x) = 1$ and $|R| \geq 2$, or $\deg(x) = 2$ and $|R| \geq 3$, then $\overline{G}$ has at most $2n-6$ edges.

Beginning with x and working clockwise around the outside of the convex hull, label the vertices of $\overline{G}$ by $0, \cdots, n-1$. Since $\overline{M}'$ is a maximal outerplanar graph drawing, it contains an edge between each pair of consecutive vertices on the outside of its convex hull, and these edges cross no other edges of $\overline{M}'$. Since $\overline{G}$ has crossing edges but $\overline{G}'$ does not, at least one edge incident with x crosses another edge of $\overline{G}$ (a blue edge of $\overline{M}$). This edge has label xj for some $j \neq 0$. Since xj crosses an edge of $\overline{G}$, j cannot be 1 or $n-1$.

Let i be the largest vertex label smaller than j to which j is adjacent in $\overline{M}'$. Let k be the smallest vertex label larger than j to which j is adjacent in $\overline{M}'$. Since j is adjacent to both $j-1$ and $j+1$, i and k exist as described. If there is no edge from i to k then $\overline{M}'$ has a face that is not a triangle – a contradiction to the maximal outerplanarity of $\overline{M}'$. Then ik is an edge and since $x = 0 < i < j < k$ the edges xj and ik cross in $\overline{M}$.

Since ijk is a triangle and xj crosses ik the paths $xjki$ and $xjik$ are self-intersecting paths of length 3 in $\overline{M}$. But $\overline{G}$ has no self-intersecting path of length 3, so either both ij and jk are red edges (Case 1), or ik is a red edge (Case 2).

Case 1: Suppose that ij and jk are red edges.

Notice that either x has degree 1 (Case 1a), or the second edge incident to x crosses no edge of $\overline{M}$ (Case 1b), or the second edge incident to x does cross an edge of $\overline{M}$ (Case 1c). These are examined below.

Case 1a: Suppose the degree of x is 1. Since we are assuming $\overline{M}$ has at least 2 red edges this is enough.

Case 1b: Suppose the second edge incident to x, say xb, crosses no edge of $\overline{M}$. In particular then xb does not cross $1(n-1)$, so b is either 1 or $n-1$. However, xj does cross $1(n-1)$ in $\overline{M}$. Then xb creates a self-intersecting path of length 3 with xj and $1(n-1)$. But this self-intersecting path cannot be monochromatically blue and we know both xb and xj are blue. Therefore $1(n-1)$ must be red – our third red edge.

Case 1c: Suppose that xb crosses an edge of $\overline{M}$. As before, this means that $\overline{M}'$ contains a triangle abc where $0 < a < b < c$. This provides two self-intersecting paths of length 3 in $\overline{M}$ and so either both ab and bc are red edges or ac is a red edge.

Notice that we do not get both $ab = jk$ and $bc = ij$ (or we would have $a = b$ and $i = k$). So if both ab and bc are red edges $\overline{M}$ has a total of at least 3 red edges.

Notice that if $ac = ij$ then xb crosses $ik = ak$ and $xbik$ is a self-intersecting path of length 3. Then either ik or ab is also a red edge. Similarly if $ac = jk$. Thus if ac is a red edge, $\overline{M}$ has at least 3 red edges.

Case 2: Suppose that ik is a red edge. By hypothesis xj crosses some blue edge, say rs. Then $x < r < j < s$. Since j is adjacent to each i and k, and rs cannot cross ij or jk, we can conclude that $r \leq i, k \leq s$ and at least one of these inequalities is strict. Choose rs to be the blue edge crossing xj that is as close as possible to ik in the sense that it minimizes $\min\{i-r, s-k\}$. A careful examination shows that either there is a red edge crossing xj that is closer to ik, or $i = r$ and ij is a red edge, or $s = k$ and jk is red. Thus we find a second red edge. An analysis entirely similar to that of Cases 1b and 1c yields a third red edge.

So if x has degree 1 then $\overline{M}$ has at least 2 red edges, and if x has degree 2 then $\overline{M}$ has at least 3 red edges. Thus $\overline{G}$ contains at most $2n-6$ edges. □

With this lemma proved we may proceed with the statement and proof of the theorem.

Theorem 2. *If $\overline{G}$ is a locally outerplanar graph with n vertices and at least one crossing then $\overline{G}$ has at most $2n-6$ edges. This bound is sharp.*

Proof. It is enough to consider the case where $\overline{G}$ is connected.

Proof by induction. Our base case is a self-intersecting path of length 4. No other edges can be added to this graph without losing local outerplanarity. The graph has 5 vertices and $2 \cdot 5 - 6 = 4$ edges. This verifies our base case and tell us that our bound, once proved, is sharp.

Let $\overline{G}$ be a locally outerplanar graph with n vertices and at least one crossing. By Theorem 1, we are guaranteed that $\overline{G}$ has a vertex x of degree at most 2. If $\overline{G} \setminus \{x\}$ has crossing edges then by induction it has at most $2(n-1) - 6$ edges and thus $\overline{G}$ has at most $2n - 6$ edges.

If $\overline{G} \setminus \{x\}$ contains no crossings, then $\overline{G} \setminus \{x\}$ is an outerplanar graph drawing. Then by Lemma 1, $\overline{G}$ has at most $2n - 6$ edges. □

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