BOIJ-SÖDERBERG THEORY AS AN INTRODUCTION TO RESEARCH IN COMMUTATIVE ALGEBRA

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Joint Mathematics Meetings 2019

AMS Special Session on Commutative Ring Theory: Research for Undergraduate and Early Graduate Students

EXPERIENCE

- As faculty supervising undergraduates
- As a graduate student working with undergraduates
- As a graduate student starting research in commutative algebra
- As an undergraduate doing mathematical research with faculty supervision

BOIJ-SÖDERBERG THEORY

Let $S = \mathbb{k}[x_1, x_2, \ldots, x_d]$ (standard graded $\mathbb{k}$-algebra over a field $\mathbb{k}$).

Let M be a finitely generated graded S-module with minimal graded free resolution

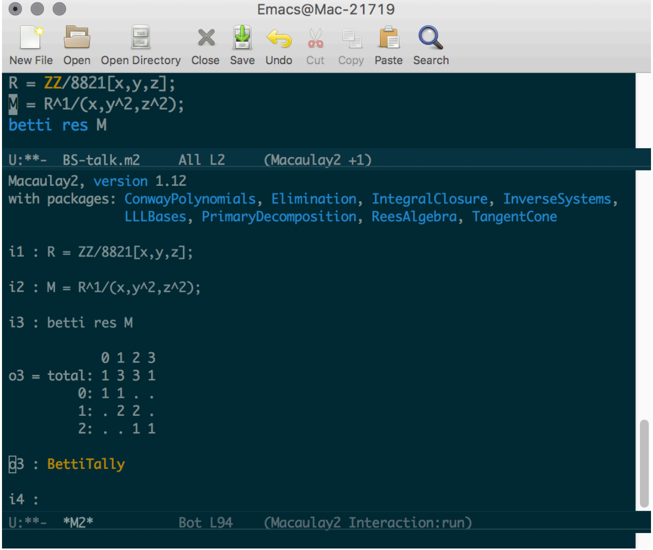
$$F_\cdot : 0 \leftarrow \oplus_j S(-j)^{\beta_{0,j}(M)} \leftarrow \oplus_j S(-j)^{\beta_{1,j}(M)} \leftarrow \cdots \leftarrow \oplus_j S(-j)^{\beta_{d,j}(M)} \leftarrow 0,$$

where $\beta_{i,j}(M)$ is the number of minimal degree j generators of $\mathrm{Syz}_i(M)$.

The **Betti diagram** of an S-module M tabulates the ranks of the free modules in the free resolution of M:

$$\beta(M) = \begin{array}{c|cccccc} & 0 & 1 & \cdots & i & \cdots & n \\ \hline 0 & \beta_{0,0}(M) & \beta_{1,1}(M) & \cdots & \beta_{i,i}(M) & \cdots & \beta_{n,n}(M) \\ 1 & \beta_{0,1}(M) & \beta_{1,2}(M) & \cdots & \beta_{i,i+1}(M) & \cdots & \beta_{n,n+1}(M) \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ j & \beta_{0,j}(M) & \beta_{1,1+j}(M) & \cdots & \beta_{i,i+j}(M) & \cdots & \beta_{j,n+j}(M) \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \end{array} .$$

COMPUTING BETTI TABLES



An S-module M is called **pure** if there is a **degree sequence** $\mathbf{d} = (d_0 < d_1 < \cdots < d_n)$ such that $\beta_{i,j}(M) = 0$ if $j \neq d_i$.

Given a degree sequence $\mathbf{d}$, define $\pi(\mathbf{d})$ to be the Betti diagram of the* pure module M with the associated degree sequence $\mathbf{d} = (d_0 < d_1 < \cdots < d_n)$ with a (technical) scaling factor. Given two degree sequences $\mathbf{c}$ and $\mathbf{d}$, we say $\mathbf{c} \leq \mathbf{d}$ if $c_i \leq d_i$ for each i.

Theorem (Boij–Söderberg ($n \leq 2$), Eisenbud–Schreyer (all n))

For every S-module M, there exists a unique list of totally ordered degree sequences $\mathbf{d}^{(1)} < \cdots < \mathbf{d}^{(r)}$ so that

$$\beta(M) = \sum q_i \pi\left(\mathbf{d}^{(i)}\right)$$

where $q_i \in \mathbb{Q}_{\geq 0}$.

Furthermore, given $\beta(M)$, there is a (fast) decomposition algorithm for determining q_i, $\mathbf{d}^{(i)}$, and the **elimination order**. `BoijSoederberg.m2`

Example ($M = S/(x, y^2, z^2)$)

$$\beta(M) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & 1 & 1 & . & . \\ 1: & . & 2 & 2 & . \\ 2: & . & . & 1 & 1 \end{array}$$

$$= 8 \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & \frac{1}{15} & \frac{1}{8} & . & . \\ 1: & . & . & \frac{1}{12} & . \\ 2: & . & . & . & \frac{1}{40} \end{array} + 8 \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & \frac{1}{30} & . & . & . \\ 1: & . & \frac{1}{6} & \frac{1}{6} & . \\ 2: & . & . & . & \frac{1}{30} \end{array} + 8 \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0: & \frac{1}{40} & . & . & . \\ 1: & . & \frac{1}{12} & . & . \\ 2: & . & . & \frac{1}{8} & \frac{1}{15} \end{array}$$

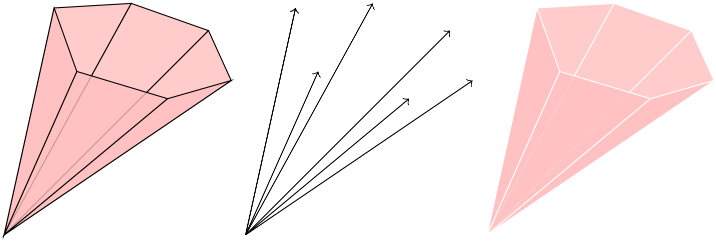
$$= 8 \cdot \pi\,(0,1,3,5) \qquad + 8 \cdot \pi\,(0,2,3,5) \qquad + 8 \cdot \pi\,(0,2,4,5)\,.$$

Elimination Table:

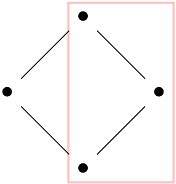
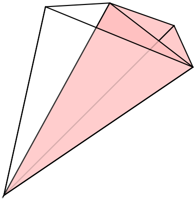
	0	1	2	3
0:	3	1	.	.
1:	.	3	2	.
2:	.	.	3	3

CONES

CONVEX CONES



Posets and Simplicial Cones

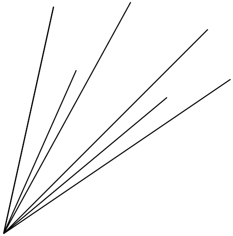
 embeds into $\mathbb{V}$ in a “good” way: 

Let $\mathbb{V}$ be the $\mathbb{Q}$-vector space of (column finite) infinite matrices $(a_{i,j+i})$.

Definition

Define the cone of Betti diagrams of finitely generated S-modules to be

$$B_{\mathbb{Q}}(S) := \left\{ \sum_{\substack{M \text{ fg} \\ S\text{-mod}}} a_M \beta(M) \,\middle|\, \begin{array}{c} a_M \in \mathbb{Q}_{\geq 0}, \\ \text{almost all } a_M \text{ are zero} \end{array} \right\} \subseteq \mathbb{V}.$$



Goal

Describe $B_{\mathbb{Q}}(S)$.

EXAMPLES OF EARLY CAREER RESEARCH PROJECTS

The Betti diagram of a complete intersection module $M = S/(f_1, \ldots, f_d)$ over the ring S is determined by the degrees of its minimal generators.

Example

If $S = \mathbb{k}[x]$ and $M = S/(f)$, then $\beta(M) = \deg(f) \cdot \pi\,(0, \deg(f))$.

If $S = \mathbb{k}[x, y]$ and $M = S/(f, g)$, then

$$\begin{aligned}\beta(M) = \deg(f)\deg(g) \cdot \pi\,(0, \deg(f), \deg(f) + \deg(g)) \\ + \deg(f)\deg(g) \cdot \pi\,(0, \deg(g), \deg(f) + \deg(g))\,.\end{aligned}$$

Question

For $S = \mathbb{k}[x_1, \ldots, x_d]$ and any complete intersection M where

$$\beta(M) = q_1\pi\left(\mathbf{d}^{(1)}\right) + \cdots + q_r\pi\left(\mathbf{d}^{(r)}\right),$$

is there a uniform formula for determining q_j and $\mathbf{d}^{(j)}$ in terms of $\deg(f_i)$?

Proposition (G–Jeffries–Mayes-Tang–Raicu–Stone–White 2015, MSRI graduate workshop)

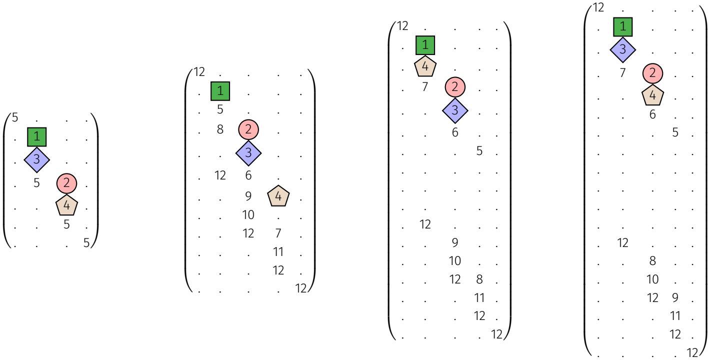
Let S be $\mathbb{k}[x_1, x_2, x_3]$, and let $I = (f_1, f_2, f_3)$ be an ideal generated by a homogeneous regular sequence with $\deg(f_i) = a_i$ where $a_i \leq a_{i+1}$ for all i. Then

$$\begin{aligned}\beta(S/I) = \quad & a_1a_2(a_2+a_3)\cdot\pi\left(0, a_1, a_1+a_2, a_1+a_2+a_3\right)\\ &+ a_1a_2(a_3-a_1)\cdot\pi\left(0, a_2, a_1+a_2, a_1+a_2+a_3\right)\\ &+ 2a_1a_2(a_1+a_3-a_2)\cdot\pi\left(0, a_2, a_1+a_3, a_1+a_2+a_3\right)\\ &+ a_1a_2(a_3-a_1)\cdot\pi\left(0, a_3, a_1+a_3, a_1+a_2+a_3\right)\\ &+ a_1a_2(a_2+a_3)\cdot\pi\left(0, a_3, a_2+a_3, a_1+a_2+a_3\right).\end{aligned}$$

Question

Does the decomposition behave uniformly for all d?

Some Elimination Tables:



$I = (x^2, y^3, u^4)$ $\quad$ $I_6 = I + (v^6)$ $\quad$ $I_{11} = I + (v^{11})$ $\quad$ $I_{13} = I + (v^{13})$

Theorem (G-Huben*-Stone 2017)

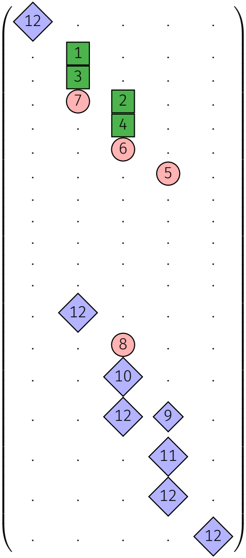
Let $c > 1$. Consider a complete intersection

$$R = \Bbbk[x_1, \dots, x_c, x_{c+1}]/(x_1^{a_1}, \dots, x_c^{a_c}, x_{c+1}^{a_{c+1}}).$$

There is an inductive algorithm for decomposing $\beta_{c+1} = \beta(a_1, \dots, a_c, a_{c+1})$ into a sum of pure diagrams (with rational coefficients, not necessarily positive) that starts with the Boij-Söderberg decomposition of β_c.

Furthermore, when $a_{c+1} \gg 0$, the new inductive algorithm produces the same output as the Boij-Söderberg algorithm.

EXAMPLE



- Phase 1: Calculate the decomposition of β_c and form coefficients for first third of β_{c+1}. Eliminate entries of β_{c+1} according to elimination order of β_c.
- Phase 2: Eliminate entires left to right along the columns.
- Phase 3: Finish eliminating the diagram using the dual of the pure diagrams from the first half of the algorithm.

Consider $\beta = \beta(a_1, \ldots, a_c, a_{c+1})$ with a_{c+1} large enough. Then

1. the recursive algorithm produces the original BS decomposition;
2. the elimination order of β is compatible with that of $\beta(a_1, \ldots, a_c)$;
3. the elimination order of β stabilizes;
4. the coefficients from Phase 1 and Phase 3 are given by linear polynomials in a_{c+1};
5. the number of terms in the BS decomposition of β is constant.

Theorem (Annunziata* G Hawkins* Sullivan* 2014, WVC REU)

A ray of the sub-cone generated by complete intersections is extremal if and only if the Betti diagram of a complete intersection lies on that ray.

Furthermore, there is a factorial time algorithm for decomposing a ray in the cone into extremal rays.

ADVICE TO ADVISORS

1. Be ambitious for your students....
2. ...but have a plan for the project.
3. Build research skills into your courses.
4. Don't do the project for your students.
5. Have regular research meetings.
6. Your best local collaborators may be your students. (-SL)

http://people.hamilton.edu/cgibbons/files/research/
research-preliminary-tasks.pdf