

When are centralizers of finite subgroups of $\text{Out}(F_n)$ finite?

Debra L. Boutin

ABSTRACT. This paper gives necessary and sufficient conditions for the centralizer of a finite subgroup of outer automorphisms (resp. automorphisms) of a finitely generated free group to be finite. If G is realized by its action on a graph, Krstić's work produces generators of the centralizer of G in terms of graph isomorphisms and certain graph transformations. This paper examines the behavior of these graph transformations to determine criteria for a centralizer to be finite. This result extends that of Krstić and Vogtmann on finite centralizers. It also gives alternate criteria to that provided by Pettet for a free-by-finite group to have a finite outer automorphism group.

1. Introduction

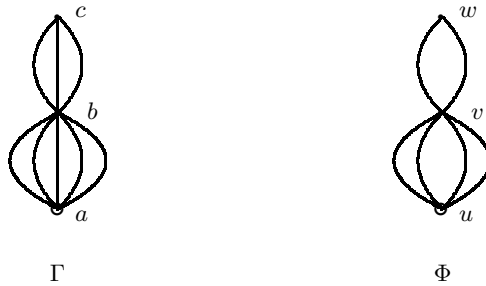


FIGURE 1

Consider the automorphism γ of the graph Γ in Figure 1, where γ cyclically permutes the edges connecting vertices a and b as well as the edges connecting vertices b and c . When considered as an automorphism of $\pi_1(\Gamma, a)$, γ has a finite centralizer. However, the automorphism of $\pi_1(\Phi, u)$ induced by the analogous graph automorphism ϕ of Φ , (ϕ cyclically permutes the edges connecting vertices u and v as well as the edges connecting vertices v and w) has an infinite centralizer. The purpose of this paper is to describe easy-to-check criteria for distinguishing these cases.

There are a number of reasons to study centralizers of finite groups of outer automorphisms. Brown [1, page 261] showed that in order to compute the integral Euler characteristic of $\text{Out}(F_n)$ it is sufficient to know the rational Euler characteristics of centralizers of finite order elements of $\text{Out}(F_n)$. Smillie and Vogtmann [8]

Partially supported by a National Defense Science and Engineering Graduate Fellowship.

gave an effective method for computing the rational Euler characteristic of $\text{Out}(F_n)$ (which can be thought of as the centralizer of the trivial automorphism) and we can hope to apply similar techniques to compute the rational Euler characteristic of the more general centralizers and thus find the integral Euler characteristic of $\text{Out}(F_n)$.

Pettet [7], in completing the classification of the finitely generated groups whose full automorphism groups are virtually free, recently provided necessary and sufficient conditions for the outer automorphism group of a free-by-finite group E to be finite. If E is given by the short exact sequence $1 \rightarrow F_n \rightarrow E \rightarrow K \rightarrow 1$, the conjugation action of E on F_n induces a homomorphism $\theta : K \rightarrow \text{Out}(F_n)$. Since $\text{Out}(E)$ is finite if and only if the centralizer of $\theta(K)$ is finite, the results in this paper provide alternate criteria for determining the finiteness of $\text{Out}(E)$. Pettet obtains results in terms of the action of E on a tree T , and equivalently, in terms of the graph of groups determined by the quotient T/E . In this paper, we obtain results in terms of the $\theta(K)$ -graph T/F_n .

To learn about the finiteness properties of automorphism groups of free-by-finite groups, Krstić and Vogtmann [6] studied the finiteness properties of centralizers of finite subgroups of $\text{Out}(F_n)$ by creating a space on which a given centralizer acts with finite stabilizers and finite quotient. From their results we can easily derive criteria for, and a description of, a finite centralizer $C(G)$ when a finite subgroup $G < \text{Out}(F_n)$ has a realization in which all edge stabilizers are normal in their associated vertex stabilizers; that is: we can derive the statement of Theorem 3.4 for outer automorphisms. Using methods different from those of [6], in Theorem 3.4 we find analogous results for this specific type of subgroup in both $\text{Out}(F_n)$ and in $\text{Aut}(F_n)$. We then extend the result in Theorem 4.3 to criteria for a general finite subgroup of $\text{Out}(F_n)$ or $\text{Aut}(F_n)$ to have a finite centralizer.

This paper is organized as follows: In Section 2 we will briefly look at necessary background definitions and ideas; we will also sketch the definitions and results of Krstić that will be used in this work. In Section 3 we state and prove Theorem 3.4, while in Section 4 we state and prove Theorem 4.3.

2. Foundations

2.1. Background and Definitions. We shall adopt the standard combinatorial definition of a *graph* [9, page 91] Γ . The set of *vertices* of Γ is denoted $V(\Gamma)$ while the set of *oriented edges* of Γ is denoted as $E(\Gamma)$. The oriented edge e is incident with two vertices, $\iota(e)$, $\tau(e)$, called the respectively the *initial* and *terminal* vertices of e . The involution $e \mapsto \bar{e}$ takes e to its inverse edge \bar{e} . Then $\iota(\bar{e}) = \tau(e)$ and $\tau(\bar{e}) = \iota(e)$. An *edge path* is a concatenation of oriented edges $e_1 \cdots e_k$ so that $\tau(e_i) = \iota(e_{i+1})$. An edge path is said to be *freely reduced* if for each i , $e_{i+1} \neq \bar{e}_i$. Any edge path can be freely reduced by deleting such canceling pairs. The *edge length* of an edge path is the number of edges in the path once it is freely reduced. A *G-graph* is a graph with an action of G on it. We will write this as a left action. A *pointed G-graph* is a G -graph where each element of G fixes the *basepoint*, which we will usually denote by $*$. A *graph isomorphism* is a bijective map $\varphi : \Gamma_1 \rightarrow \Gamma_2$ that takes vertices to vertices, edges to edges and is equivariant with respect to the incidence relation τ (and therefore ι) on oriented edges. That is, $\tau(e) = \tau(f)$ in the graph Γ_1 if and only if $\tau(\varphi(e)) = \tau(\varphi(f))$ in the graph Γ_2 . An *isomorphism of*

(pointed) G -graphs is a graph isomorphism that is G -equivariant (and that takes the basepoint of one to the basepoint of the other).

We shall assume, as is usual in geometric group theory, that no element of G inverts any edge of the graph. If necessary, we could add a vertex to the center of each inverted edge to avoid the problem. The *orbit* of $x \in E(\Gamma) \cup V(\Gamma)$, denoted $\mathcal{O}(x)$, is the set of all images of x under the action of G . We will denote the set of images of e and its inverse edge \bar{e} , by $\mathcal{O}(e, \bar{e})$, and will often consider this as a subgraph of Γ . A *reduced* G -graph is a G -graph in which each edge orbit, $\mathcal{O}(e, \bar{e})$, when considered as a graph, has non-trivial fundamental group. Equivalently, a G -graph is reduced if it contains no G -invariant forests.

If G is a subgroup of A then the centralizer of G in A is the set of all $a \in A$ for which $aga^{-1} = g$ for all $g \in G$. Thus in order to talk about the centralizer of G , it is essential to know in what larger group we are working. If $G \leq \text{Out}(F_n)$, $C(G)$ is the centralizer of G in $\text{Out}(F_n)$ and if $G \leq \text{Aut}(F_n)$, $C(G)$ is the centralizer of G in $\text{Aut}(F_n)$. We will work in $\text{Out}(F_n)$ throughout this paper, with the modifications necessary for $\text{Aut}(F_n)$ contained in parentheses. Denote by $p : \text{Aut}(F_n) \rightarrow \text{Out}(F_n)$ the canonical projection map.

Both Culler [2] and Zimmermann [10] have independently proved that if G is a finite subgroup of $\text{Out}(F_n)$ then there is a G -graph Γ and an identification $\pi_1(\Gamma) \leftrightarrow F_n$ so that the induced action of G on $\pi_1(\Gamma)$ is identical to the action of G on F_n as a group of outer automorphisms. If G is a finite subgroup of $\text{Aut}(F_n)$ we can assume that the graph Γ is a pointed G -graph and that the induced action of G on $\pi_1(\Gamma, *)$ is identical to the action of G on F_n as a group of automorphisms. In each of these cases, we say that G is *realized* by its action on Γ .

2.2. Krstić's Theorem. In [4], Krstić proved that if we choose the “right” kind of graph realizing G , we can read off the generators of the centralizer of G from the graph. In $\text{Out}(F_n)$ the “right” kind of graph is a reduced G -graph; in $\text{Aut}(F_n)$ it is a pointed reduced G -graph. If G is realized by the (pointed) reduced G -graph Γ and φ is a (basepoint preserving) G -equivariant automorphism of Γ , then the outer automorphism (automorphism) of $\pi_1(\Gamma, *)$ induced by φ commutes with the action of G on $\pi_1(\Gamma, *)$. That is, φ induces an element of $C(G)$. Thus we can easily find some elements of the centralizer by looking at Γ , but since Γ is finite, we can only find finitely many elements in this way.

Krstić defines another way of G -equivariantly transforming graphs that leads to more elements of the centralizer. If the reduced (pointed) G -graph Γ contains oriented edges a and b so that $\tau(a) = \tau(b)$, $a \notin \mathcal{O}(b, \bar{b})$ and $\text{stab}(a) \subseteq \text{stab}(b)$, we can transform Γ into a new graph by “sliding” the terminal points of edges in $\mathcal{O}(a)$ along appropriate edges in $\mathcal{O}(b)$. More specifically, the *Nielsen transformation* $N = \langle a, b \rangle$ transforms Γ into the (pointed) reduced G -graph $N(\Gamma)$ which has the same edge and vertex sets as Γ and whose incidence relation differs only on terminal points of images of a . If $g(a)$ is an image of a then $\tau(g(a)) = \tau(g(b))$ in Γ while $\tau(g(a)) = \iota(g(b))$ in $N(\Gamma)$. The graphs Γ and $N(\Gamma)$ are not necessarily isomorphic, but they do have G -equivariantly isomorphic fundamental groups. In particular, there is a G -equivariant isomorphism of edge path groupoids $\Pi(\Gamma) \xrightarrow{\langle a, b \rangle} \Pi(N(\Gamma))$ defined on generators by

$$\langle a, b \rangle : \begin{cases} g(a) & \mapsto g(a)g(b) & \forall g \in G \\ g(\bar{a}) & \mapsto g(\bar{b})g(\bar{a}) & \forall g \in G \\ c & \mapsto c & \forall c \notin \mathcal{O}(a, \bar{a}) \end{cases} .$$

Note that since $\text{stab}(a) \subseteq \text{stab}(b)$ this is well defined. As an isomorphism of path groupoids, $\langle a, b \rangle$ takes a path between vertices v and u in Γ to a path between v and u in $N(\Gamma)$. Note that a Nielsen transformation does not move vertices, so in particular, it does not move the basepoint. Thus by restriction to edge loops at a basepoint, $\langle a, b \rangle$ gives us a G -equivariant isomorphism of fundamental groups.

EXAMPLE 2.1. In Figure 2 let the generator of $\mathbb{Z}_3 \cong G < \text{Aut}(F_4)$ act on Γ by cyclic permutation of the edges connecting vertices v and w (the orbit of a) and simultaneously by cyclic permutation of the edges connecting vertices v and u (the orbit of b). The stabilizers of a and b are trivial and since $\tau(a) = \tau(b)$ we get a Nielsen transformation $N = \langle a, b \rangle$ yielding the pointed G -graph $N(\Gamma)$. Notice that a is an edge path from w to v in Γ while $N(a) = ab$ is an edge path from w to v in $N(\Gamma)$.

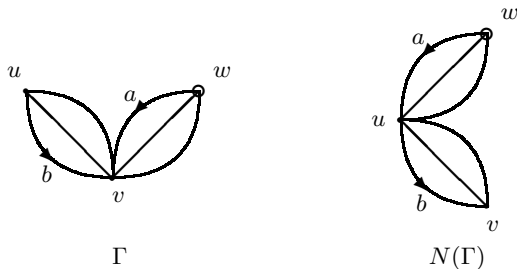


FIGURE 2

We shall use the notation $\langle a, b \rangle$ to denote the graph transformation, the isomorphism of path groupoids, and the isomorphism of fundamental groups. This should cause no confusion in context. We will consider Nielsen transformations to act on the left. Then a product of Nielsen transformations $\langle a_k, b_k \rangle \cdots \langle a_2, b_2 \rangle \langle a_1, b_1 \rangle$ takes the graph Γ to Γ_k through a sequence of Nielsen transformations:

$$\Gamma_0 \xrightarrow{\langle a_1, b_1 \rangle} \Gamma_1 \xrightarrow{\langle a_2, b_2 \rangle} \cdots \xrightarrow{\langle a_k, b_k \rangle} \Gamma_k.$$

Notice that the product of Nielsen transformations $\langle a, b_k \rangle \cdots \langle a, b_2 \rangle \langle a, b_1 \rangle$ makes sense precisely when $b_k \cdots b_2 b_1$ is an edge path terminating at $\tau(a)$ that is fixed edgewise by $\text{stab}(a)$ and this transformation can be written as $\langle a, b_k \cdots b_2 b_1 \rangle$. Any product of Nielsen transformations taking Γ to another graph is called a *Nielsen product applicable to Γ* . If Γ and Γ' are reduced G -graphs realizing G and there is a product of Nielsen transformations taking one to another, up to a G -equivariant isomorphism of graphs, then Γ and Γ' are called *Nielsen equivalent*. By Krstić [4, Theorem 2] all reduced G -graphs realizing G are Nielsen equivalent.

Krstić's main results [4, Propositions 4 and 4'] show that for any element f in the centralizer of G there is a product of Nielsen transformations N taking Γ to a Nielsen equivalent graph $N(\Gamma)$, and a (basepoint preserving) G -equivariant graph isomorphism $\phi : N(\Gamma) \rightarrow \Gamma$ so that $\phi_* N_* = f$. In other words, every element of the centralizer of G is induced by a Nielsen product applicable to Γ followed by a (basepoint preserving) G -equivariant graph isomorphism.

If a reduced (pointed) G -graph Γ realizing G has no Nielsen transformations applicable to it, the centralizer of G is generated solely by (basepoint preserving) G -equivariant graph automorphisms of Γ . Since Γ is finite, there are only finitely

many such automorphisms, and therefore $C(G)$ is finite. The following example illustrates this situation.

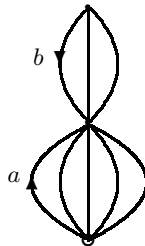


FIGURE 3

EXAMPLE 2.2. Let $G \cong \mathbb{Z}_{15} < \text{Aut}(F_6)$ and $p(G) < \text{Out}(F_6)$. Note that $G \cong p(G)$ and denote the generator of each of these groups by g . Let g act on the graph Γ in Figure 3 by cyclic permutation of the bottom edges (the orbit of a), and by cyclic permutation of the top edges (the orbit of b). Then each of G and $p(G)$ is realized by its action on Γ . The stabilizer of a (and therefore of all of $\mathcal{O}(a, \bar{a})$) is $\langle g^5 \rangle$; the stabilizer of b is $\langle g^3 \rangle$. Since $\text{stab}(a) \not\subset \text{stab}(b)$ and $\text{stab}(b) \not\subset \text{stab}(a)$, there are no Nielsen compatible edges in Γ . Thus $C(p(G)) < \text{Out}(F_6)$ is isomorphic to the group of G -equivariant graph automorphisms of Γ and $C(G) < \text{Aut}(F_6)$ is isomorphic to the group of basepoint preserving G -equivariant graph automorphisms of Γ . Thus $C(p(G))$ and $C(G)$ are both finite.

What if we do have Nielsen transformations? What if we have a whole “loop” of them? Suppose that $\beta = b_k \cdots b_1$ is a non-trivial edge loop outside of $\mathcal{O}(a, \bar{a})$ based at $\tau(a)$ and stabilized edgewise by $\text{stab}(a)$. Then $N = \langle a, \beta \rangle$ is a Nielsen product applicable to Γ , $N(\Gamma) = \Gamma$, and N induces a G -equivariant automorphism of the path groupoid and of the fundamental group of Γ . More precisely, N takes the path a in Γ to the path $a\beta$, and N^2 takes a to $a\beta^2$. Since β is a non-trivial edge loop the edge length of $aN^j(a)$ grows as j grows. Thus N induces an infinite order automorphism of the path groupoid of Γ . How does N behave on the fundamental group of Γ ? It is possible that both N_* , the induced automorphism on $\pi_1(\Gamma, *)$, and $p(N_*)$, its projection into $\text{Out}(F_n)$, have infinite order (c.f. Example 2.3); it is possible that N_* is the identity and therefore so is $p(N_*)$ (c.f. Example 2.4); and it is possible that N_* is an inner automorphism of F_n (c.f. Example 2.5). The following examples illustrate these cases.

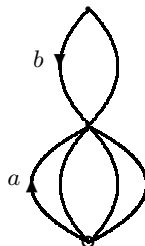


FIGURE 4

EXAMPLE 2.3. Let $G \cong \mathbb{Z}_4 < \text{Aut}(F_4)$. Note that $G \cong p(G)$ and denote the generator of each of these groups by g . Let g act on the graph Γ in Figure 4 by cyclic permutation of the bottom edges (the orbit of a) and by cyclic permutation of the top edges (the orbit of b). The stabilizer of a (and therefore of all of $\mathcal{O}(a, \bar{a})$) is $\{1\}$; the stabilizer of b is $\langle g^2 \rangle$. Let β be the edge loop $\bar{b}g(b)$. Since $\text{stab}(a)$ fixes each edge of β and β is an edge loop, $N = \langle a, \beta \rangle$ is a product of Nielsen transformations taking Γ to itself. Noting that $g(\bar{\beta}) = \beta$, we get that N takes the edge loop $ag(\bar{a})$ to the edge loop $a\beta^2g(\bar{a})$, and more generally that $N^k(ag(\bar{a})) = a\beta^{2k}g(\bar{a})$. Since clearly the edge length of this loops grows as k does, N_* has infinite order. Further, since $a \neq g(a)$, $N^k(ag(\bar{a}))$ is cyclically reduced so there is no k so that N^k is an inner automorphisms and therefore $p(N_*)$ also has infinite order. Thus, in this case, both $C(G)$ and $C(p(G))$ are infinite.

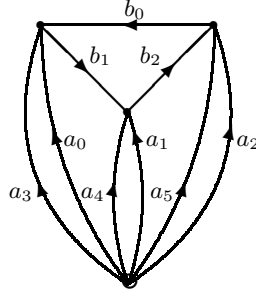


FIGURE 5

EXAMPLE 2.4. Let $G \cong \mathbb{Z}_6 < \text{Aut}(F_6)$. Note that $G \cong p(G)$ and denote the generator of each of these groups by g . Let g act on the graph Γ in Figure 5 by $g : \begin{cases} a_i & \mapsto a_{i+1 \pmod{6}} \\ b_i & \mapsto b_{i+1 \pmod{3}} \end{cases}$. For ease of notation, denote a_0 by a and b_0 by b . The stabilizer of a (and therefore of all of $\mathcal{O}(a, \bar{a})$) is $\{1\}$; the stabilizer of b is $\langle g^3 \rangle$. Let β denote the edge loop $g(b)g^2(b)b$. Since $\text{stab}(a)$ fixes each edge of β and β is a loop, $N = \langle a, \beta \rangle$ is a Nielsen product taking Γ to itself. Consider the effect of N on the edge loops $ag^3(\bar{a})$ and $ag(b)g^4(\bar{a})$.

$$N(ag^3(\bar{a})) = a\beta g^3(\bar{\beta})g^3(\bar{a}) = a\beta\bar{\beta}g^3(\bar{a}) = ag^3(\bar{a});$$

$$\begin{aligned} N(ag(b)g^4(\bar{a})) &= a\beta g(b)g^4(\bar{\beta})g^4(\bar{a}) \\ &= ag(b)g^2(b)bg(b)g(\bar{b})\bar{b}g^2(\bar{b})g^4(\bar{a}) = ag(b)g^4(\bar{a}). \end{aligned}$$

Since all edge loops in $\pi_1(\Gamma, *)$ are concatenations of loops of the above types, we could show that the effect of N on every edge loop of Γ based at $*$ is trivial. Thus each of N_* and $p(N_*)$ is the identity. Here both $C(G)$ and $C(p(G))$ are finite.

EXAMPLE 2.5. Let $G \cong \mathbb{Z}_2 < \text{Aut}(F_2)$. Note that $G \cong p(G)$ and denote the generator of each of these groups by g . Let g act on the graph Γ in Figure 6 so that g fixes the edge b and transposes the two edges between $\tau(a)$ and $\iota(a)$ (the orbit of a). Since the stabilizer of a fixes the edge loop b the Nielsen transformation



FIGURE 6

$N = \langle a, b \rangle$ takes Γ to itself. N induces an inner automorphism of $\pi_1(\Gamma, *)$ which is conjugation by the element of F_2 associated with the edge loop b . For example, the edge loop $\bar{a}g(a)$ becomes $\bar{b}\bar{a}g(a)b$ under N . Thus N induces an infinite order element of $\text{Aut}(F_n)$ but induces the identity element in $\text{Out}(F_n)$. In this situation, $C(G)$ is infinite while $C(p(G))$ is finite.

The following section gives results which differentiate these cases.

3. Groups with Normal Edge Stabilizers

Let G be a finite subgroup of $\text{Out}(F_n)$ (or $\text{Aut}(F_n)$) and let Γ be a (pointed) reduced G -graph realizing G .

DEFINITION 3.1. An oriented edge a of Γ is called *Nielsen compatible* if there is a Nielsen transformation $\langle a, b \rangle$ applicable to Γ .

In Example 2.4 we showed a graph which had a Nielsen compatible edge and the Nielsen products taking Γ back to itself that arose from moving this edge had trivial effect on $\pi_1(\Gamma)$. In this situation we wish to call the edge, and all associated Nielsen transformations, “ π_1 -trivial.” This term is defined below and in Proposition 3.6 we prove that this is the definition we want.

DEFINITION 3.2. An oriented edge a of Γ is said to be π_1 -trivial if

1. it is Nielsen compatible,
2. $\mathcal{O}(\tau(a))$ is a proper subset of $V(\Gamma)$ (and the basepoint is outside $\mathcal{O}(\tau(a))$), and
3. the connected component of $\Gamma \setminus \mathcal{O}(a, \bar{a})$ that contains $\tau(a)$ is a simple edge loop that can be written as $g(b) \cdots g^r(b)b$ for some oriented edge b , some element $g \in G$, and some integer r .

EXAMPLE 3.3. Let $\langle g \rangle = \mathbb{Z}_6$ and consider the \mathbb{Z}_6 -subgraph in Figure 7. Since $\text{stab}(a) = \{1\} \subset \langle g^3 \rangle = \text{stab}(b)$, a is a Nielsen compatible edge. Since $\Gamma \setminus \mathcal{O}(a, \bar{a})$ is the simple edge loop $g(b)g^2(b)b$, and there are vertices outside of $\mathcal{O}(\tau(a))$, a is a π_1 -trivial edge.

Whenever $\langle a, b \rangle$ is a Nielsen transformation applicable to Γ such that a is a π_1 -trivial edge, then we call $\langle a, b \rangle$ a π_1 -trivial Nielsen transformation and the ordered pair (a, b) a π_1 -trivial pair. Note that, as illustrated in Figure 7, in this situation $\mathcal{O}(b, \bar{b})$ is the only edge orbit incident with $\mathcal{O}(a, \bar{a})$ at the vertices in $\mathcal{O}(\tau(a))$.

Krstić and Vogtmann obtained results on the finiteness of $C(G)$ for a particular type of finite subgroups of $\text{Out}(F_n)$. Theorem 3.4 extends this result to the same type of subgroups of $\text{Aut}(F_n)$. This “type” of subgroup is one which can be realized by a (pointed) reduced G -graph Γ in which each edge stabilizer is normal in its associated vertex stabilizers. In this case G will be said to be *realized normally* by

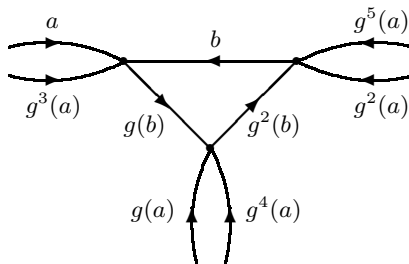


FIGURE 7

Γ . If b is an edge whose edge stabilizer is normal in its associated vertex stabilizers, then there is an edge loop β contained in $\mathcal{O}(b, \bar{b})$ that is fixed by $\text{stab}(b)$. As a consequence of this, when G is realized normally by Γ and $\langle a, b \rangle$ is any Nielsen transformation applicable to Γ , then there is a Nielsen product $\langle a, \beta \rangle$, with β as above, taking Γ to itself. This fact is crucial in 2) \implies 3) of Theorem 3.4 below.

THEOREM 3.4. *Let G be a finite subgroup of $\text{Out}(F_n)$ (or $\text{Aut}(F_n)$) that is realized normally by the (pointed) reduced G -graph Γ . Then the following are equivalent.*

1. $C(G)$ is isomorphic to the group of (basepoint preserving) G -equivariant graph automorphisms of Γ ;
2. $C(G)$ is finite;
3. Every Nielsen compatible edge of Γ is π_1 -trivial.

PROOF. 1. \implies 2.

The group of (basepoint preserving) G -equivariant automorphisms of Γ is a subgroup of $\text{Aut}(\Gamma)$, which is finite since Γ is finite.

2. \implies 3.

The following proposition is the heart of this part of the proof. Note that this proposition does not require that G be realized normally; this proposition will be used again later to prove criteria for general finite subgroups of $\text{Out}(F_n)$ (or $\text{Aut}(F_n)$) to have a finite centralizer.

PROPOSITION 3.5. *Let $G < \text{Out}(F_n)$ (or $\text{Aut}(F_n)$) be finite with a finite centralizer; let Γ be a (pointed) reduced G -graph realizing G . If Γ has a Nielsen compatible edge a for which there exists an edge loop β at $\tau(a)$ that is outside of $\mathcal{O}(a, \bar{a})$ and is fixed by $\text{stab}(a)$, then a is π_1 -trivial.*

PROOF. Let a and β be as described in the statement of the proposition. Then $N = \langle a, \beta \rangle$ is a Nielsen transformation applicable to Γ . Without loss of generality, we may assume β is a simple edge loop (or go back and choose one that is). Write $\beta = b_1 \cdots b_r b$ as a reduced edge path. Let C be the connected component of $\Gamma \setminus \mathcal{O}(a, \bar{a})$ that contains b , and therefore $\tau(a)$.

Claim: $\mathcal{O}(\tau(a))$ is a proper subset of $V(\Gamma)$.

If $\mathcal{O}(\tau(a)) = V(\Gamma)$ then $\tau(a) = \iota(g(a))$ for some $g \in G$. Then there is an integer s so that $\sigma = a g(a) \cdots g^s(a)$ is an edge loop. Further the basepoint is in $\mathcal{O}(\tau(a))$ and without loss of generality we may assume that the basepoint is $\iota(a)$. Then σ is an edge loop at the basepoint of Γ . Applying powers of N to σ yields: $N^k(\sigma) = a \beta^k g(a) g(\beta^k) \cdots g^s(a) g^s(\beta^k)$. Clearly this edge loop grows in edge length as k grows, so N_* has infinite order. Further, since β is outside $\mathcal{O}(a, \bar{a})$, each $N^k(\sigma) = a \beta^k g(a) g(\beta^k) \cdots g^s(a) g^s(\beta^k)$ is cyclically reduced. Thus $p(N_*)$ also has infinite order.

Thus if $\mathcal{O}(\tau(a))$ is not a proper subset of $V(\Gamma)$, $C(G)$ is infinite. So we may assume $\mathcal{O}(\tau(a))$ is properly contained in $V(\Gamma)$.

Claim: If $G < \text{Aut}(F_n)$, the basepoint is not in C .

Since C is connected and $\tau(a)$ is a vertex of C , if the basepoint is also there then there exists an edge path ψ within C from the basepoint to $\tau(a)$. Since as a subgraph, $\mathcal{O}(a, \bar{a})$ has non-trivial fundamental group, we may choose ϕ to be an edge path in $\mathcal{O}(a, \bar{a})$ from $\iota(a)$ to $\tau(a)$ that is not a itself. Then $\sigma = \psi \bar{a} \phi \bar{\psi}$ is a non-trivial, freely reduced edge loop at the basepoint. Consider the effect of N^k on σ . Since ψ is in C , it contains no edges of $\mathcal{O}(a, \bar{a})$ and therefore N^k leaves it unchanged, $N^k(\bar{a}) = \bar{\beta}^k \bar{a}$, and $N^k(\phi)$ we'll just write as $N^k(\phi)$. Then $N^k(\sigma) = \psi \bar{\beta}^k \bar{a} N^k(\phi) \bar{\psi}$. The edge length of $\psi \bar{\beta}^k$, the edge path that precedes the first occurrence of \bar{a} in $N^k(\sigma)$, increases as k increases. Thus there is no integer k so that $N^k(\sigma) = \sigma$ and therefore N_* has infinite order.

Thus, since $C(G)$ is finite, we may assume that the basepoint is not in C . A similar argument shows that the basepoint is not in any image of C , and so we may conclude that the basepoint is outside $\mathcal{O}(\tau(a))$.

Then for every edge $h(a) \in \mathcal{O}(a)$ there is a (possibly trivial) edge path outside $\mathcal{O}(a, \bar{a}) \cup \mathcal{O}(C)$ from the base point to $\iota(h(a))$. Let $\psi_{h(a)}$ denote such a path.

Claim: C is a simple edge loop.

We may consider the simple edge loop β as a subgraph of C . If β is a proper subgraph of C there is an oriented edge and its inverse $\{e, \bar{e}\}$ in $C \setminus \beta$. Since Γ is a reduced G -graph, $\mathcal{O}(e, \bar{e})$ has non-trivial fundamental group, so we may choose an edge loop at $\iota(e)$ that contains e . Call this loop γ' . Use edges of β as necessary to complete γ' to an edge loop based at $\tau(a)$ and call it γ . Note that since neither e nor \bar{e} is an edge of β , γ and β are independent elements of $\pi_1(C, \tau(a))$. In particular, $\beta^k \gamma \bar{\beta}^k$ and γ always represent distinct edge loops. Consider the effect of powers of N on the edge loop $\sigma_1 = \psi_a a \gamma \bar{a} \bar{\psi}_a$: for every positive integer k , $N^k(\sigma_1) = \psi_a a \beta^k \gamma \bar{\beta}^k \bar{a} \bar{\psi}_a$ which is unequal to σ_1 by the above. Thus N_* has infinite order.

Note that if $w \in F_n$ so that $\sigma_1 = w \sigma_1 \bar{w}$, then $w = \psi_a a \beta^k \bar{a} \bar{\psi}_a$. Since Γ is a reduced G -graph and there is no edge of $\mathcal{O}(a)$ whose initial point is $\tau(a)$, there must be an edge $g(a) \in \mathcal{O}(a)$, distinct from a , that shares the terminal point of a . Then $\sigma_2 = \psi_a a g(\bar{a}) \bar{\psi}_{g(a)}$ is an edge loop at the basepoint and $N^k(\sigma_2) = \psi_a a \beta^k g(\bar{\beta}^k) g(\bar{a}) \bar{\psi}_{g(a)}$. Note that $N^k(\sigma_2) \neq w \sigma_2 w^{-1}$, in particular because on the right hand side $g(\bar{a})$ cannot cancel with a , so $g(\bar{a}) \bar{\psi}_{g(a)}$ cannot cancel with $\psi_a a$. Thus there is no integer k for which N_*^k is an inner automorphism and so $p(N_*)$ also has infinite order.

Thus if C is not a simple edge loop, $C(G)$ is infinite. So we may assume that β and C are the same as subgraphs of Γ and therefore that C is a simple edge loop.

Claim: $\beta = g(b) \cdots g^r(b) b$ for some oriented edge b , some element $g \in G$.

We have that $\beta = b_1 b_2 \dots b_r b$ as a freely reduced edge loop based at $\tau(a)$. Since Γ is a reduced G -graph, b is contained in a loop in its orbit. This loop is a subgraph of β and therefore must be β itself. Then each edge of β is in the orbit of b and in particular, there exists $g \in G$ so that either $b_1 = g(b)$ or $b_1 = g(\bar{b})$.

Suppose that $b_1 = g(\bar{b})$. Then g fixes $\tau(b)$ and acts as a reflection of β over the vertex $\tau(b)$. That is, $g(\beta) = \bar{\beta}$. Since $\text{stab}(a) \subseteq \text{stab}(b)$ and g does not fix b , g does not fix a . Thus $g(a)$ is an edge distinct from a that also terminates at $\tau(b)$. Let $\sigma = \psi_a a g(\bar{a}) \bar{\psi}_{g(a)}$, an edge loop at the basepoint. Then $N^k(\sigma) = \psi_a a \beta^k g(\bar{\beta})^k g(\bar{a}) \bar{\psi}_{g(a)} = \psi_a a \beta^{2k} g(\bar{a}) \bar{\psi}_{g(a)}$. We can see that the edge length of $N^k(\sigma)$ increases as k increases, and thus that N_* has infinite order. Since $g(a) \neq a$ there is no $w \in F_n$ so that $w \psi_a a g(\bar{a}) \bar{\psi}_{g(a)} \bar{w} = \psi_a a \beta^{2k} g(\bar{a}) \bar{\psi}_{g(a)}$, so $p(N_*)$ also has infinite order.

Since $C(G)$ is finite we may assume that $b_1 \neq g(\bar{b})$ and therefore that $b_1 = g(b)$. Then $bg(b)$ is an edge path and so is $g^i(b)g^{i+1}(b)$ for every integer i . Since Γ is finite, this means that there is an integer r so that $bg(b)g^2(b) \cdots g^r(b)$ is a simple edge loop at $\iota(b)$. Rewriting this as an edge loop at $\tau(b)$, and recalling that β is the connected component of $\Gamma \setminus \mathcal{O}(a, \bar{a})$ containing $\tau(b)$, we can conclude that β can be written as $g(b)g^2(b) \cdots g^r(b)b$.

Thus a is π_1 -trivial. □

Since G is realized normally by Γ , every Nielsen compatible edge of Γ meets the hypotheses of Proposition 3.5. Thus if $C(G)$ is finite, every Nielsen compatible edge of Γ is π_1 -trivial.

Before we get to the last portion of this proof, we need to learn more about the subgraph $\mathcal{O}(a, \bar{a}) \cup \mathcal{O}(b, \bar{b})$ when the pair (a, b) is π_1 -trivial. The facts gathered here do not require that the edge stabilizers of G be normal in their vertex stabilizers and will be used throughout the rest of the paper.

- Since C is a connected component of $\Gamma \setminus \mathcal{O}(a, \bar{a})$, the orbit of C is the union of disjoint copies of C . Since each element of G must take C to an isomorphic copy of itself, each $g \in G$ either takes C to itself (an automorphism of C) or takes C completely off itself.

- If we consider C to be an $(r+1)$ -gon, its automorphism group is D_{r+1} the dihedral group of order $2(r+1)$. It is generated by a rotation of order $r+1$ (which we can think of as g here), and a reflection. If there is an element $h \in G$ which reflects C over some vertex $\tau(g^i(b))$ then $h(g^i(b)) = g^{i+1}(\bar{b})$ and therefore $g^{-i-1}hg^i$ sends b to \bar{b} . Since G acts on Γ without inversions, this cannot happen. Thus there is no element of G that acts as a reflection of C . Then every automorphism of C in G is equivalent to a power of g in the sense that if $h \in G$ and $h(C) = C$ then $h|_C = g^j|_C$ for some integer j .

- If an element $h \in G$ fixes a vertex of C then by the above $h|_C = g^j|_C$. But the only power of g that fixes a vertex of C is $g^0 = 1$. Thus if any element of G fixes a vertex of C , it fixes C itself. This also tells us that within each image of C all edge stabilizers and vertex stabilizers are the same.

- If $h, k \in G$ so that $h(a)$ and $k(a)$ have the same terminal point, then $k^{-1}h$ fixes $\tau(a)$ and therefore C . Thus $h|_C = k|_C$. In particular, $h(b) = k(b)$.

• Let $f \in G$ be arbitrary. Since C can be written as $g(b) \cdots g^r(b) b$, $f(C)$ can be written as $fg(b) \cdots fg^r(b) f(b)$. Thus fgf^{-1} rotates the cycle $f(C)$.

Now we may complete the proof of Theorem 3.4.

3. \implies 1.

Suppose that all Nielsen compatible edges of Γ are π_1 -trivial. Since Γ is a reduced (pointed) G -graph we have an injective map from the group of (basepoint preserving) G -equivariant graph automorphisms of Γ to $C(G)$. So to show that $C(G)$ is isomorphic to this group, it is enough to show that for every $f \in C(G)$ there exists a (basepoint preserving) G -equivariant graph automorphism φ so that $f = p(\varphi_*)$ on $\pi_1(\Gamma)$ (or $f = \varphi_*$ on $\pi_1(\Gamma, *)$). Recall that by Krstić [4, Propositions 4 and 4'], if $f \in C(G)$ then $f = p(\phi_* N_*)$ (or $\phi_* N_*$) where N is a Nielsen product applicable to Γ and ϕ is an isomorphism of (pointed) G -graphs. Thus it is enough to show that for every Nielsen product N applicable to Γ there exists a (basepoint preserving) G -equivariant graph isomorphism, $\varphi : \Gamma \rightarrow N(\Gamma)$, so that N and φ induce the same isomorphism of fundamental groups. The following proposition proves this without requiring that all edge stabilizers of G be normal in their vertex stabilizers, or that all Nielsen compatible edges of Γ be π_1 -trivial. This proposition will be used again later to obtain results for general finite subgroups of $\text{Out}(F_n)$ (or $\text{Aut}(F_n)$).

PROPOSITION 3.6. *Let G be a finite subgroup of $\text{Out}(F_n)$ (or $\text{Aut}(F_n)$) and Γ a (pointed) reduced G -graph realizing G .*

1. *Let a be an oriented edge that is π_1 -trivial and $N = \langle a, b \rangle$ a Nielsen transformation applicable to Γ . Then the (basepoint preserving) isomorphism of G -graphs given by*

$$\varphi : \begin{cases} h(b) & \mapsto hg^{-1}(b) & \forall h \in G \\ c & \mapsto c & \forall c \notin \mathcal{O}(b, \bar{b}) \end{cases}$$

induces the same isomorphism of fundamental groups as N does.

2. *If N is a product of π_1 -trivial Nielsen transformations so that $N(\Gamma) = \Gamma$, then N induces the trivial outer automorphism (automorphism) on the fundamental group.*

PROOF. Let a be an oriented edge that is π_1 -trivial and $N = \langle a, b \rangle$ a Nielsen transformation applicable to Γ . Since a is π_1 -trivial either the basepoint is outside $\mathcal{O}(\tau(a))$ (automorphism case) or there is some vertex outside of $\mathcal{O}(\tau(a))$ that we may choose as a basepoint (outer automorphism case). Further, there is an element $g \in G$ and an integer r so that we may write the connected component of $\mathcal{O}(b, \bar{b})$ that contains b as $g(b) \cdots g^r(b) b$.

Let $h_1(a) h_2(b) h_3(\bar{a})$ be an edge path in $\mathcal{O}(a, \bar{a}) \cup \mathcal{O}(b, \bar{b})$. Recall that $\psi_{h_1(a)}$ and $\psi_{h_3(a)}$ are edge paths outside of $\mathcal{O}(a, \bar{a}) \cup \mathcal{O}(b, \bar{b})$ from the basepoint to $\iota(h_1(a))$ and $\iota(h_3(a))$ respectively. Then $\sigma = \psi_{h_1(a)} h_1(a) h_2(b) h_3(\bar{a}) \bar{\psi}_{h_3(a)}$ is an edge loop at the basepoint. Note that $\tau(h_1(a)) = \iota(h_2(b))$ since σ is an edge path. Further, $\iota(h_2(b)) = \tau(h_2g^{-1}(b))$ and $\tau(h_2g^{-1}(b)) = \tau(h_2g^{-1}(a))$ so we can conclude that $h_1(b) = h_2g^{-1}(b)$. In an entirely similar manner we can show that $h_3(b) = h_2(b)$.

Now we can compute:

$$\begin{aligned} N(\psi_{h_1(a)} h_1(a) h_2(b) h_3(\bar{a}) \bar{\psi}_{h_3(a)}) &= \\ \psi_{h_1(a)} h_1(a) h_1(b) h_2(b) h_3(\bar{b}) h_3(\bar{a}) \bar{\psi}_{h_3(a)} &= \\ \psi_{h_1(a)} h_1(a) h_2 g^{-1}(b) h_3(\bar{a}) \bar{\psi}_{h_3(a)} & \end{aligned}$$

Thus the effect of N on a loop of the form σ can be mimicked by relabeling the edge $h_2(b)$ as $h_2 g^{-1}(b)$. Since N induces a homomorphism, its effect on the inverse form of σ , $\psi_{h_3(a)} h_3(a) h_2(\bar{b}) h_1(\bar{a}) \bar{\psi}_{h_1(a)}$ can be mimicked by relabeling the edge $h_2(\bar{b})$ as $h_2 g^{-1}(\bar{b})$. The reader can verify that all loops in $\pi_1(\Gamma, *)$ can be written as concatenations of loops of the form σ and $\bar{\sigma}$ so the effect of N_* on $\pi_1(\Gamma, *)$ can be duplicated by relabeling each edge $h(b^\pm)$ as $h g^{-1}(b^\pm)$. Further, it is easy to check that the map $\varphi : \begin{cases} h(b) & \mapsto & h g^{-1}(b) & \forall h \in G \\ c & \mapsto & c & \forall c \notin \mathcal{O}(b, \bar{b}) \end{cases}$ represents a (basepoint preserving) G -equivariant graph isomorphism from Γ to $N(\Gamma)$.

Thus a single π_1 -trivial Nielsen transformation applicable to Γ can have its effect on fundamental groups mimicked by an isomorphism of (pointed) G -graphs. Now let's look at the case of a π_1 -trivial Nielsen product.

After performing the π_1 -trivial Nielsen transformation $\langle a, b \rangle$ we may perform the π_1 -trivial Nielsen transformation $\langle a, g^{-1}(b) \rangle$. Again this induces the same isomorphism of fundamental groups as the isomorphism of G -graphs which sends b to $g^{-1}(b)$ and fixes all edges outside of $\mathcal{O}(b, \bar{b})$. The composition of these two Nielsen transformations then induces the same isomorphism of fundamental groups as the isomorphism of G -graphs which sends b to $g^{-2}(b)$. Continuing this line of reasoning we see that the product of π_1 -trivial Nielsen transformations $\langle a, g(b) \cdots g^r(b) b \rangle$ induces the same automorphism of $\pi_1(\Gamma, *)$ as the isomorphism of G -graphs that takes b to $g^{-(r+1)}(b) = b$. That is, $N = \langle a, g(b) \cdots g^r(b) b \rangle$ induces the identity automorphism on $\pi_1(\Gamma, *)$.

Suppose that $N = \langle a, \beta \rangle$ is a product of π_1 -trivial Nielsen transformations so that $N(\Gamma) = \Gamma$. Then since the terminal point of a returns to its original position, β must be an edge path that traverses $g(b) \cdots g^r(b) b$, in a positive or negative direction, some integral number of times. By the above then, N induces the identity on $\pi_1(\Gamma, *)$.

Now suppose that N is any product of π_1 -trivial Nielsen transformations so that $N(\Gamma) = \Gamma$. If $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$ are π_1 -trivial Nielsen transformations and $a_1 \neq a_2$ as oriented edges, then $\mathcal{O}(b_1, \bar{b}_1) \cap \mathcal{O}(b_2, \bar{b}_2) = \emptyset$ so by Krstić's relation (R3) in [5], these two Nielsen transformations commute. That is, when $a_1 \neq a_2$ and both are π_1 -trivial, $\langle a_1, b_1 \rangle \langle a_2, b_2 \rangle = \langle a_2, b_2 \rangle \langle a_1, b_1 \rangle$. So we may rearrange the factors of N so that all factors moving $\tau(a_1)$ come first, all factors moving $\tau(a_2)$ come afterward, etc. That is, we may write N as $\langle a_k, \beta_k \rangle \cdots \langle a_1, \beta_1 \rangle$ so that $a_i \neq a_j$ if $i \neq j$. Further, since every vertex is returned to its original position after N is performed and $\tau(a_i)$ is only moved by $\langle a_i, \beta_i \rangle$, each β_i must be an edge loop based at $\tau(a_i)$.

Note that in the automorphism case, if N is a product of Nielsen transformations taking Γ to Γ , and $N = \langle a_1, \beta_1 \rangle \cdots \langle a_k, \beta_k \rangle$ as above, then by definition the basepoint is outside $\cup_i \mathcal{O}(\tau(a_i))$. Then by our previous work, each $N_i = \langle a_i, \beta_i \rangle$ induces the identity automorphism on $\pi_1(\Gamma, *)$ so the product must also.

However, in the outer automorphism case though we know that there are vertices outside $\mathcal{O}(\tau(a_i))$ for each i , it is possible that $\cup_i \mathcal{O}(\tau(a_i)) = V(\Gamma)$ (c.f. Example 2.5). In this case, we can compute the outer automorphism induced by each $\langle a_i, \beta_i \rangle$, separately. For each $\langle a_i, \beta_i \rangle$ use a basepoint from outside $\mathcal{O}(\tau(a_i))$, and we get the identity automorphism on $\pi_1(\Gamma, *)$ which projects to the identity automorphism in $\text{Out}(F_n)$, where we no longer need worry about the basepoint. Thus the product of the projections is trivial, so the projection of the product, $p(N_*)$, is the identity in $\text{Out}(F_n)$. \square

Using Proposition 3.6, we can conclude that if all the Nielsen compatible edges of Γ are π_1 -trivial, the effect of every Nielsen product can be duplicated by an isomorphism of (pointed) G -graphs. Thus $C(G)$ is isomorphic to the group of (basepoint preserving) G -equivariant graph automorphisms of Γ . \square

4. General Finite Subgroups

What can occur when G is not normally realized by Γ , the reduced (pointed) G -graph realizing it? Let a be an arbitrary oriented edge of Γ and let \mathcal{F}_a be the connected component of $\Gamma \setminus \mathcal{O}(a, \bar{a})$ that is fixed by $\text{stab}(a)$ and contains $\tau(a)$. Whenever we have a path β in \mathcal{F}_a that terminates at $\tau(a)$ we can move $\tau(a)$ along that path by the Nielsen product $\langle a, \beta \rangle$. By Proposition 3.5, if β happens to be an edge loop and a is not π_1 -trivial, $\langle a, \beta \rangle$ induces an infinite order element of the centralizer. However, if G is not realized normally, \mathcal{F}_a may be non-trivial but may not contain an edge loop. In this case Nielsen transformations may provide us with elements in the centralizer other than those coming from (basepoint preserving) G -equivariant graph automorphisms, but they might not produce an infinite centralizer. In this section we study what happens when \mathcal{F}_a contains no loops, and find criteria for a centralizer of a finite subgroup of $\text{Out}(F_n)$ (or $\text{Aut}(F_n)$) to be finite.

DEFINITION 4.1. Let $G < \text{Out}(F_n)$ (or $\text{Aut}(F_n)$) be a finite subgroup realized by the reduced (pointed) G -graph Γ . Let a be an oriented edge of Γ . Let Γ_a be the maximal connected subgraph of Γ that contains a and is fixed by $\text{stab}(a)$. Let \mathcal{A} be the connected component of $\Gamma_a \cap \mathcal{O}(a, \bar{a})$ that contains a . Let \mathcal{F}_a be the connected component of $\Gamma_a \setminus \mathcal{O}(a, \bar{a})$ that contains $\tau(a)$.

Since we will also be working with graphs that are Nielsen equivalent to Γ , we need notation for the analogous sets in $N(\Gamma)$. Denote by $N(\Gamma_a)$ the maximal connected subgraph of $N(\Gamma)$ that contains a and is fixed by $\text{stab}(a)$; denote by $N(\mathcal{A})$ the connected component of $N(\Gamma_a) \cap \mathcal{O}(a, \bar{a})$ that contains a ; denote by $N(\mathcal{F}_a)$ the connected component of $N(\Gamma_a) \setminus \mathcal{O}(a, \bar{a})$ that contains $\tau(a)$. One can show that as edge sets $\Gamma_a = N(\Gamma_a)$. However, because Nielsen transformations can change the way in which edge orbits are incident, we may have $\mathcal{F}_a \neq N(\mathcal{F}_a)$ and/or $\mathcal{A} \neq N(\mathcal{A})$.

DEFINITION 4.2. We say that an edge of Γ *deadends* if for every Nielsen product N applicable to Γ , $N(\mathcal{F}_a)$ is a tree.

Note that if a is not Nielsen compatible in Γ , then \mathcal{F}_a consists solely of the vertex $\tau(a)$. Therefore if a is not Nielsen compatible in any graph which is Nielsen equivalent to Γ , a deadends. Example 4.6 shows an instance of the more general case in which there are Nielsen compatible edges that deadend.

THEOREM 4.3. *Let $G < \text{Out}(F_n)$ (or $\text{Aut}(F_n)$) be finite and be realized by the (pointed) reduced G -graph Γ . Then $C(G)$ is finite if and only if every oriented edge of Γ is either π_1 -trivial or deadend.*

PROOF. \Leftarrow

Let Γ be a (pointed) reduced G -graph realizing G in which all oriented edges of Γ are either π_1 -trivial or deadend. Before we begin this proof, we need to consider the types of Nielsen transformations that can occur when we start with such a graph.

- Suppose there is a π_1 -trivial edge c in $\Gamma_a \setminus \mathcal{O}(a, \bar{a})$. Then $\langle c, d \rangle$ is a Nielsen transformation applicable to Γ and d is in $\mathcal{F}_a \cup \mathcal{O}(a, \bar{a})$. If $d \notin \mathcal{O}(a, \bar{a})$ then \mathcal{F}_a contains edges from both $\mathcal{O}(c, \bar{c})$ and $\mathcal{O}(d, \bar{d})$ and thus a cannot be π_1 -trivial. Since c is π_1 -trivial, $\mathcal{O}(d, \bar{d})$ is the disjoint union of edge cycles so \mathcal{F}_a is not a tree and a does not deadend. Since this contradicts our hypothesis, if a is not the second component of a π_1 -trivial pair then there is no π_1 -trivial edge in \mathcal{F}_a . We may generalize result to: if neither a nor its inverse is the second component of a π_1 -trivial pair in any $N(\Gamma)$ then there are no π_1 -trivial edges in $\Gamma_a \setminus \mathcal{O}(a, \bar{a})$.

- Let a be a π_1 -trivial edge of Γ . If either a or \bar{a} is the second component of a Nielsen transformation $\langle c, d \rangle$ applicable to Γ , the subgraph $\mathcal{A} \cup \mathcal{F}_a$ is contained in the connected component of $\Gamma_c \setminus \mathcal{O}(c, \bar{c})$ that contains $\tau(c)$. Since a is π_1 -trivial, \mathcal{F}_a is the union of disjoint edge loops, and thus c is neither π_1 -trivial nor deadend. Since this contradicts our hypotheses, if a is a π_1 -trivial edge of Γ , neither a nor its inverse is the second component of a Nielsen transformation applicable to Γ .

- Let a be a π_1 -trivial edge of Γ . Suppose that there is some Nielsen transformation N so that a is not π_1 -trivial in $N(\Gamma)$. Since every Nielsen transformation having a as a first component preserves the π_1 -triviality of a , then either N is a Nielsen transformation $\langle c, a \rangle$ which moves a new orbit into the connected component of $N(\Gamma) \setminus \mathcal{O}(a, \bar{a})$ that contains $\tau(a)$, or N is a Nielsen transformation $\langle g(b), a \rangle$ or $\langle g(\bar{b}), a \rangle$ which breaks up the cycle $g(b) \cdots g^r(b) b$. But by our above work the first case cannot occur, and since a is π_1 -trivial $\text{stab}(b) \not\subseteq \text{stab}(a)$ and the second case cannot occur. Thus if a is π_1 -trivial in Γ and all edges of Γ are either π_1 -trivial or deadend, then a is π_1 -trivial in every graph that is Nielsen equivalent to Γ . Further, we may generalize the previous conclusion to: if a is a π_1 -trivial edge of Γ , neither a nor its inverse is the second component of a Nielsen transformation applicable to any $N(\Gamma)$.

- Let (a, b) be a π_1 -trivial pair. If a shows up as the first component of of a Nielsen transformation $\langle a, d \rangle$ applicable to some graph $N(\Gamma)$ and $d \notin \mathcal{O}(b, \bar{b})$ then both $\mathcal{O}(b, \bar{b}) \cap N(\mathcal{F}_a)$ and $\mathcal{O}(d, \bar{d}) \cap N(\mathcal{F}_a)$ are non-trivial. Since we've shown that this cannot happen, whenever (a, b) is a π_1 -trivial pair and $\langle a, d \rangle$ is a Nielsen transformation applicable to some $N(\Gamma)$, then $d \in \mathcal{O}(b, \bar{b})$.

In this first part of the proof we will see that when N is a product of Nielsen transformations taking Γ back to itself the effect of N is that of a π_1 -trivial Nielsen product. That is, as an isomorphism of path groupoids $N = T$ where T is a product of π_1 -trivial Nielsen transformations. By Proposition 3.6, this will tell us that N induces the trivial element. We will then use this result to show that whenever we have a graph that is Nielsen equivalent to Γ , there are a finite number of elements of the centralizer that come from Nielsen transformations taking Γ to Γ' . To get started we need the following definition.

DEFINITION 4.4. Let a be an oriented edge of Γ . We say that a Nielsen transformation $N = \langle c, d \rangle$ is *from inside* Γ_a if there is some $g \in G$ so that both $g(c)$ and $g(d)$ are in Γ_a . Otherwise we say that N is *from outside* Γ_a .

First we will show that when N is a Nielsen product taking Γ to itself and a is an oriented edge of Γ , there are Nielsen products $T_a, T_{\bar{a}}, I$ and O where $T_{\bar{a}}$ is a product of π_1 -trivial Nielsen transformations that move $\tau(\bar{a})$ (possibly a trivial product), T_a is a product of π_1 -trivial Nielsen transformations that move $\tau(a)$ (possibly a trivial product), I is a product of Nielsen transformations from inside Γ_a that are not in T_a or $T_{\bar{a}}$, O is a product of Nielsen transformation from outside Γ_a , and $N = T_{\bar{a}} T_a I O$.

Notice that if $\langle c, d \rangle$ is from outside Γ_a then for each $g(c)$ either $\text{stab}(a)$ does not fix $g(c)$ or there is no edge path from $\tau(a)$ to $\tau(g(c))$ that is fixed by $\text{stab}(a)$. Since $\text{stab}(g(c)) \subseteq \text{stab}(g(d))$ and $\tau(g(c)) = \tau(g(d))$ this implies that $g(d)$ is not in Γ_a . Thus if $\langle c, d \rangle$ is from outside Γ_a then both c and d are. If $g(c)$ and $g(d)$ are both in Γ_a we may as well assume that c and d are. Otherwise we could just relabel our Nielsen transformation as $\langle g(c), g(d) \rangle$.

By working through the notation we could show that Nielsen transformations $\langle c, d \rangle$ and $\langle e, f \rangle$ commute if and only if $c \notin \mathcal{O}(e)$, $c \notin \mathcal{O}(f, \bar{f})$ and $e \notin \mathcal{O}(d, \bar{d})$. This fact will be used throughout this proof.

Consider two Nielsen transformations: $\langle c, d \rangle$ from inside Γ_a , and $\langle e, f \rangle$ from outside Γ_a . Since no image of e or f is in Γ_a but both c and d are in Γ_a , $e, f \notin \mathcal{O}(c, \bar{c}), \mathcal{O}(d, \bar{d})$. Thus $\langle c, d \rangle$ and $\langle e, f \rangle$ commute. This shows that we can reorder the Nielsen transformations that comprise the product N so that all factors that are from outside Γ_a are on the right and all factors that are from inside Γ_a are on the left. That is we have Nielsen products I and O from inside and outside Γ_a respectively, so that $N = IO$.

Now we wish to rewrite I as $T_{\bar{a}} T_a I'$, where I' is a product of Nielsen transformations from inside Γ_a that that are not in T_a or $T_{\bar{a}}$.

Suppose that a is π_1 -trivial. Consider $\langle a, d \rangle$ and $\langle e, f \rangle$ where all $d, e, f \in \Gamma_a$ and $e \notin \mathcal{O}(a)$. By previous work since all edges here are either π_1 -trivial or deadend and since a is π_1 -trivial, $a \notin \mathcal{O}(f, \bar{f})$ and $e \notin \mathcal{O}(d, \bar{d})$. Thus $\langle a, d \rangle$ and $\langle e, f \rangle$ commute. The same result holds for $\langle \bar{a}, d \rangle$ and $\langle e, f \rangle$ when \bar{a} is π_1 -trivial and $e \notin \mathcal{O}(\bar{a})$. Thus π_1 -trivial Nielsen transformations moving $\tau(a)$ commute with everything else in I , except things like themselves, and π_1 -trivial Nielsen transformations moving $\iota(a)$ commute with everything else in I , except things like themselves. Thus we can rewrite I as $I = T_{\bar{a}} T_a I'$ as desired. Relabeling I' as I we have that $N = T_{\bar{a}} T_a IO$.

Our goal is to show that $N(a) = T_{\bar{a}} T_a(a)$.

If $N(a) = a$ we can let $T_{\bar{a}}$ and T_a be the identity transformation and we'll be done. Thus we can assume that $N(a) \neq a$ and in particular that neither a nor its inverse is ever the second component of a π_1 -trivial pair (since if it were neither a nor its inverse would ever show up as the first component of a Nielsen transformation and we would have $N(a) = a$).

Notice that by construction, the set of oriented edges whose terminal points can be moved by $T_{\bar{a}}, T_a, I$ and O are disjoint. Thus edges moved by O remain unmoved under $T_{\bar{a}} T_a I$ and thus must be left by O in their final position. By hypothesis, this final position is their original position. Thus $O(\Gamma) = \Gamma$, $T_{\bar{a}} T_a I(\Gamma) = \Gamma$, and since O never moves the end points of a , $O(a) = a$. Repeating this reasoning gives us that $I(\Gamma) = \Gamma$, but we'll need to prove that $I(a) = a$.

This is the heart of the proof. The idea is that since the connected components of $\Gamma_a \setminus \mathcal{O}(a, \bar{a})$ are always trees and there are no loops for $\tau(a)$ to move around, $\tau(a)$ must not move at all.

Let $I(a) = c_k \cdots c_1 a b_1 \cdots b_\ell$ so that $c_k \cdots c_1$ and $b_1 \cdots b_\ell$ are reduced edge paths. Since I induces an automorphism on the edge paths of Γ , $I(a)$ is a path from $i(a)$ to $\tau(a)$. Thus $\iota(c_k) = \iota(a)$ and $\tau(b_\ell) = \tau(a)$. Since this is an edge path $\tau(c_1) = \iota(a)$ and $\tau(a) = \iota(b_1)$. Thus $c_k \cdots c_1$ is an edge loop based at $\iota(a)$ and $b_1 \cdots b_\ell$ is an edge loop based at $\tau(a)$. Further, these edge loops are contained in Γ_a .

Case 1) If both a and \bar{a} are π_1 -trivial there are no Nielsen transformations from inside Γ_a that are not in T_a or $T_{\bar{a}}$. Thus I is trivial and $I(a) = a$. Then we may assume that not both a and \bar{a} are π_1 -trivial.

Case 2) If \bar{a} is π_1 -trivial and a deadends then

- $c_k \cdots c_1$ is trivial, since there are no Nielsen transformations in I taking $\tau(\bar{a}) = \iota(a)$ anywhere.
- If $b_1 \cdots b_\ell$ is not wholly contained in \mathcal{F}_a then, by the definition of \mathcal{F}_a , $b_1 \cdots b_\ell$ must contain an edge of $\Gamma_a \cap \mathcal{O}(a, \bar{a})$. But we've seen that since \bar{a} is π_1 -trivial there is no way to have a Nielsen transformation with an image of a or \bar{a} as its second component. Thus there is no way to get an edge of $\Gamma_a \cap \mathcal{O}(a, \bar{a})$ into $b_1 \cdots b_\ell$. Thus $b_1 \cdots b_\ell$ is a loop contained in \mathcal{F}_a which means it is trivial since \mathcal{F}_a is a tree.

The same is true if a is π_1 -trivial and \bar{a} deadends. Thus we may assume:

Case 3) Both a and \bar{a} deadend.

If there is no oriented edge b in \mathcal{F}_a or $\mathcal{F}_{\bar{a}}$ so that $\text{stab}(b) = \text{stab}(a)$, then this is also true of the images of \mathcal{F}_a and $\mathcal{F}_{\bar{a}}$ and thus there is no Nielsen transformation from inside Γ_a that has an image of a or \bar{a} as a second component. Thus $b_1 \cdots b_\ell$ and $c_k \cdots c_1$ contain no images of a or \bar{a} . Then $b_1 \cdots b_\ell$ is contained in the connected component of $\Gamma_a \setminus \mathcal{O}(a, \bar{a})$ that contains $\tau(a)$. That is $b_1 \cdots b_\ell$ is contained in the tree \mathcal{F}_a and is therefore a trivial loop. Similarly, $c_k \cdots c_1$ is a trivial loop and $I(a) = a$.

Thus we may assume that there is a b in \mathcal{F}_a or $\mathcal{F}_{\bar{a}}$ so that $\text{stab}(b) = \text{stab}(a)$. Note that since $b \in \Gamma_a$ it is not π_1 -trivial and thus must deadend. Since $\Gamma_a = \Gamma_b$, for every Nielsen transformation N applicable to Γ $N(\mathcal{A})$ is a subgraph of $N(\Gamma_b) \setminus \mathcal{O}(b, \bar{b})$ which means $N(\mathcal{A})$ is a tree.

Thus we have that for every Nielsen transformation N applicable to Γ each image of $N(\mathcal{F}_a)$, $N(\mathcal{F}_{\bar{a}})$ and $N(\mathcal{A})$ must be a tree. If Γ_a itself is a tree, then each of $c_1 \cdots c_k$ and $b_1 \cdots b_\ell$ is a trivial edge loop and $I(a) = a$. In the following I will show that if we assume Γ_a is not a tree, we can find some Nielsen transformation N so that there is a non-trivial edge loop in one of $N(\mathcal{F}_a)$, $N(\mathcal{F}_{\bar{a}})$, and $N(\mathcal{A})$, a contradiction.

Let $\sigma = \sigma_0 \sigma_1 \cdots \sigma_s$ be an edge loop in Γ_a . Note that since each image of \mathcal{F}_a and $\mathcal{F}_{\bar{a}}$ is a tree, there is at least one edge of $\mathcal{O}(a, \bar{a})$ in σ . Without loss of generality, assume that the first edge of the edge loop σ is a . If a is the only edge of $\mathcal{O}(a, \bar{a})$ in σ , let $N = \langle a, \bar{\sigma}_s \cdots \bar{\sigma}_1 \rangle$. Then a is a single edge loop in $N(\mathcal{A})$ (See Figure 8.)

Thus we may assume that a is not the only edge of $\mathcal{O}(a, \bar{a})$ in σ . Let j be the smallest positive integer so that $\sigma_j \in \mathcal{O}(a, \bar{a})$. If $\sigma_j = g(a)$ for some $g \in G$, let

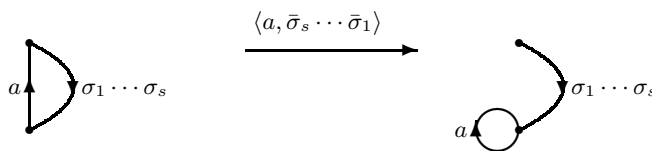


FIGURE 8

$N = \langle a, \bar{\sigma}_{j-1} \cdots \bar{\sigma}_1 \rangle$. Then in $N(\Gamma)$, $\tau(a) = \iota(g(a))$, and there is an integer t so that $\tau(a) = \tau(g^t(a))$. Thus $a g(a) \cdots g^t(a)$ is an edge loop of $N(\mathcal{A})$. (See Figure 9.)

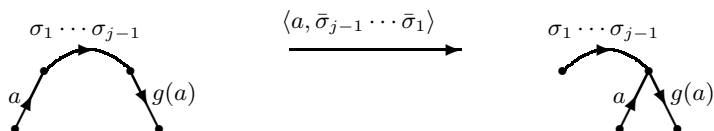


FIGURE 9

Thus we may assume that $\sigma_j = g(\bar{a})$. Denote $\sigma_1 \cdots \sigma_{j-1}$ by α , an edge path from $\tau(a)$ to $\tau(g(a))$ that is contained in \mathcal{F}_a . There is some integer t so that $\tau(g^t(a)) = \tau(a)$ which means that $\alpha g(\alpha) \cdots g^t(\alpha)$ is an edge loop in \mathcal{F}_a . (See Figure 10.)

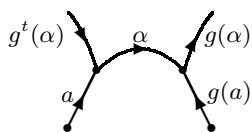


FIGURE 10

Thus we may assume that α is trivial and $\sigma = a g(\bar{a}) \sigma_2 \cdots \sigma_s$. If a and $g(\bar{a})$ are the only two edges of $\mathcal{O}(a, \bar{a})$ in σ , then denote $\bar{\sigma}_s \cdots \bar{\sigma}_2$ by β , an edge path from $i(a)$ to $i(g(a))$ that is contained in $\mathcal{F}_{\bar{a}}$. There is some integer t so that $\iota(g^t(a)) = \iota(a)$ which implies that $\bar{\beta} \cdots g^t(\bar{\beta})$ is an edge loop in $\mathcal{F}_{\bar{a}}$. (See Figure 11.)

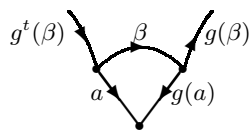


FIGURE 11

Thus we may assume that β is trivial which implies that $\sigma = ag(\bar{a})$ an edge loop in \mathcal{A} .

Thus we may assume that there are more than two edges of $\mathcal{O}(a, \bar{a})$ in σ . In the following the ideas are much the same as above and so figures will not be furnished.

Let k be the smallest integer greater than j so that that $\sigma_k = h(a)$ or $h(\bar{a})$ for some $h \in G$. If $\sigma_k = h(\bar{a})$, let $N = \langle h(a), \sigma_2 \cdots \sigma_k \rangle$ and notice that in $N(\Gamma)$, $\tau(h(a)) = i(g(a))$. Thus $h(a)g(a)gh^{-1}g(a) \cdots (gh^{-1})^t g(a)$ is an edge loop in $N(\mathcal{A})$, for some integer t .

Thus we may assume that $\sigma_k = h(a)$. Denote $\sigma_2 \cdots \sigma_{k-1}$ by γ . Then there is some integer t so that $\tau((gh^{-1})^t h(a)) = \tau(h(a))$ and thus $\bar{\gamma} \cdots (gh^{-1})^t (\bar{\gamma})$ is an edge loop in \mathcal{F}_a .

Thus we may assume that γ is trivial. Then $\sigma = ag(\bar{a})h(a)\sigma_3 \cdots \sigma_s$. In particular, $g(\bar{a})$ is an edge path from $\tau(a)$ to $\tau(h(a))$. There is an integer t so that $\tau(h^t(a)) = \tau(a)$, and so $g(\bar{a})hg(\bar{a}) \cdots h^t g(\bar{a})$ is an edge loop in \mathcal{A} .

Thus if there is a loop σ in Γ_a , then there is an edge loop in one of $N(\mathcal{F}_a)$, $N(F_{\bar{a}})$ or $N(\mathcal{A})$ for some Nielsen transformation N . Thus Γ_a is a tree and $I(a) = a$.

Thus we have that $N(a) = T_{\bar{a}} T_a I O(a) = T_{\bar{a}} T_a(a)$.

For each positively oriented a in $E(\Gamma)$ we may write, $N(a) = T_{\bar{a}} T_a(a)$, for appropriate T_a and $T_{\bar{a}}$. When all Nielsen compatible edges are π_1 -trivial or deadend, π_1 -trivial transformations $\langle A, B \rangle$ and $\langle C, D \rangle$ commute as long as $A \neq C$ (A may even equal \bar{C} and they'll still commute). Thus we can write $N = \prod_{a \in E(\Gamma)} T_{\bar{a}} T_a$. Then by Proposition 3.6, this product of π_1 -trivial Nielsen transformations induces the identity outer automorphism (automorphism) on F_n .

Now that we know how products of Nielsen transformations taking Γ to Γ behave, let's look at what we get by composing Nielsen products with isomorphisms of pointed G -graphs, $\Gamma \xrightarrow{N} \Gamma' \xrightarrow{\varphi} \Gamma$.

Let N_1, N_2 be distinct Nielsen products taking Γ to Γ' and suppose that Γ' is isomorphic to Γ as a pointed G -graph. Let φ_1, φ_2 be isomorphisms of pointed G -graphs from Γ' to Γ . I wish to show that $\varphi_1 N_1$ and $\varphi_2 N_2$ are the same up to a (basepoint preserving) G -equivariant graph automorphism of Γ . The following lemma will help us get there.

LEMMA 4.5. *Let $\varphi : \Gamma \rightarrow \Gamma$ be a (basepoint preserving) G -equivariant graph automorphism, $\langle a, b \rangle : \Gamma \rightarrow \Gamma'$ a Nielsen transformation. Define the (basepoint preserving) G -equivariant graph automorphism $\varphi' : \Gamma' \rightarrow \Gamma'$ by $\varphi'(x) = y$ in $\Gamma' \iff \varphi(x) = y$ in Γ . Define $\langle \varphi^{-1}(a), \varphi^{-1}(b) \rangle : \Gamma \rightarrow \Gamma'$ a Nielsen transformation. Then $\varphi' \langle \varphi^{-1}(a), \varphi^{-1}(b) \rangle = \langle a, b \rangle \varphi$.*

PROOF. The idea here is that $\varphi(\Gamma)$ simply has its edges labeled differently than Γ does, but we want to perform the move prescribed by $\langle a, b \rangle$ on Γ' , on a graph, $\varphi^{-1}(\Gamma') = \Gamma$, whose labels look different. Thus we perform the Nielsen transformation $\langle \varphi^{-1}(a), \varphi^{-1}(b) \rangle$. We could check that this is a valid Nielsen transformation, and it is because φ^{-1} is a G -equivariant graph automorphism. However, the result of $\langle \varphi^{-1}(a), \varphi^{-1}(b) \rangle$ and $\langle a, b \rangle \varphi$ differ by the labeling of the edges and vertices. So let φ' be the automorphism of Γ' that relabels things in the required way. That is, $\varphi'(x) = y$ in Γ' if and only if $\varphi(x) = y$ in Γ . \square

Now consider the map $N_2^{-1}\varphi_2^{-1}\varphi_1N_1$ from Γ to Γ . Notice that $\varphi_2^{-1}\varphi_1$ is an automorphism of Γ' . By an extension of the above lemma there is a graph automorphism $(\varphi_2^{-1}\varphi_1)'$ of Γ and a Nielsen transformation $(N_2^{-1})'$ from Γ' to Γ so that $N_2^{-1}\varphi_2^{-1}\varphi_1N_1 = (\varphi_2^{-1}\varphi_1)'(N_2^{-1})'N_1$.

$$\text{Then } \left(\varphi_2N_2\right)_*^{-1} \left(\varphi_1N_1\right)_* = \left((\varphi_2^{-1}\varphi_1)'\right)_* \left((N_2^{-1})'N_1\right)_*.$$

By our previous work, since $(N_2^{-1})'N_1$ is a product of Nielsen transformations taking Γ precisely to itself, $\left((N_2^{-1})'N_1\right)_* = \text{Id}$. Thus we can conclude that

$$\left(\varphi_2N_2\right)_*^{-1} \left(\varphi_1N_1\right)_* = \left((\varphi_2^{-1}\varphi_1)'\right)_*$$

and so $(\varphi_2N_2)_*$ and $(\varphi_1N_1)_*$ differ only by an automorphism induced by a (basepoint preserving) G -equivariant graph automorphism of Γ .

Let m equal the number graph automorphisms of Γ that are (basepoint preserving and) G -equivariant.

By Krstić, every element of $C(G)$ is induced by a Nielsen product followed by an isomorphism of (pointed) G -graphs. By the above work, whenever Γ' is a graph that is Nielsen equivalent to Γ and that is also isomorphic to Γ under a isomorphism of pointed G -graphs, there are at most m elements of the centralizer coming from a product of Nielsen transformations $\Gamma \rightarrow \Gamma'$ followed by an isomorphism of (pointed) G -graphs, $\Gamma' \rightarrow \Gamma$. Thus for every graph that is both Nielsen equivalent to Γ and isomorphic to Γ as a (pointed) G -graph, we get at most m distinct automorphisms in the centralizer of G .

Since Γ is finite, there are finitely many graphs that are Nielsen equivalent to Γ and thus finitely many that are Nielsen equivalent and isomorphic to Γ as (pointed) G -graphs. Further, since Γ is finite the number of (basepoint preserving) G -equivariant automorphisms of Γ , m , is finite.

Thus $C(G)$ is finite.

\implies

Let a be an edge of Γ that is Nielsen compatible but does not deadend. Then there is some Nielsen equivalent graph Γ' in which the component of $\Gamma'_a \setminus \mathcal{O}(a, \bar{a})$ that contains $\tau(a)$ also contains a non-trivial loop β based at $\tau(a)$. Relabel Γ' as Γ . By Proposition 3.5 from the proof of Theorem 3.4, since β is a non-trivial edge loop of $\Gamma \setminus \mathcal{O}(a, \bar{a})$ based at $\tau(a)$ that is fixed edgewise by $\text{stab}(a)$, $\langle a, \beta \rangle$ induces an infinite order element of $C(G)$ unless a is π_1 -trivial.

Thus if $C(G)$ is finite, every edge of Γ either is π_1 -trivial or deadends. \square

The following is an example of a finite subgroup G , where the graph realizing G has a Nielsen compatible edge that deadends. Actually, in this case all edges deadend.

EXAMPLE 4.6. Let $G < \text{Aut}(F_4)$ be isomorphic to S_3 , the symmetric group on three elements, so that G is realized by the following action on the graph Γ in Figure 12. Let G act by all permutations on the set of edges on the right (which will then be the orbit of a), and by all permutations on the set of edges on the left (the orbit of b). The stabilizer of a is generated by the permutation that transposes the other two edges of that set. Note that $\text{stab}(a)$ is not normal in $\text{stab}(\tau(a)) = G$ and that there is exactly one edge in the orbit of b that is also stabilized by $\text{stab}(a)$; assume that this is b itself. Notice that we can slide the orbits of a and b along



FIGURE 12

each other to get graphs that are Nielsen equivalent to Γ but in each of them the subgraph of $\Gamma \setminus \mathcal{O}(a, \bar{a})$ that is fixed by $\text{stab}(a)$ is a single edge. The same is true of b and every image of either a or b . Thus every edge of Γ deadends.

Notice that $N = \langle a, b \rangle$ is a Nielsen transformation applicable to Γ and that Γ and $N(\Gamma)$ (Figure 12) are isomorphic as a pointed G -graphs. If we define $\varphi : N(\Gamma) \rightarrow G$ by $b \mapsto \bar{b}$, $a \mapsto a$ and extend G -equivariantly, we get such an isomorphism. Then N takes the loop $ag(a)$ of Γ to the loop $abg(\bar{b})g(\bar{a})$ of $N(\Gamma)$ and φ takes this to the loop $a\bar{b}g(b)g(\bar{a})$ in Γ . Thus φN does not induce the identity on $\pi_1(\Gamma, *)$ and since it takes a loop of edge length 2 to a loop of edge length 4, its effect on loops is not mimicked by any automorphism of Γ . We can check that $(\varphi N)_*$ has order two. Theorem 4.3 tells us that all elements of $C(G)$ that are not induced simply by graph automorphisms are induced in the above manner. Thus we get that $C(G)$ is finite.

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TRINITY COLLEGE, HARTFORD, CT
 E-mail address: debra.boutin@trincoll.edu