The Cost of 2-Distinguishing Selected Kneser Graphs and Hypercubes

Debra L. Boutin
Hamilton College
Clinton, NY 13323
dboutin@hamilton.edu

Abstract

A graph G is said to be 2-*distinguishable* if there is a labeling of the vertices with two labels so that only the trivial automorphism preserves the labels. The minimum size of a label class in such a labeling of G is called the *cost of* 2-*distinguishing* and is denoted by $\rho(G)$. This paper shows that $\rho(K_{2^m-1:2^{m-1}-1}) = m+1$ – the only result so far on the cost of 2-distinguishing Kneser graphs. The result for Kneser graphs is adapted to show that $\rho(Q_{2^m-2}) = \rho(Q_{2^m-1}) = \rho(Q_{2^m}) = m + 2$ – a significant improvement on previously known bounds for the cost of 2-distinguishing hypercubes.

1 Introduction

A labeling of the vertices of a graph G with the integers $1, \ldots, d$ is called a d-*distinguishing labeling* if no non-trivial automorphism of G preserves the labels. A graph is called d-*distinguishable* if it has a d-distinguishing labeling. Albertson and Collins introduced distinguishing labeling in [3]. Recent work shows that large members of many graph families are 2-distinguishable. Examples of 2-distinguishable finite graphs include hypercubes Q_n with $n \geq 4$ [4], Cartesian powers G^n for a connected graph $G \neq K_2, K_3$ and $n \geq 2$ [1, 12, 14], Kneser graphs $K_{n:k}$ with $n \geq 6, k \geq 2$ [2], and (with seven small exceptions) 3-connected planar graphs [9]. Examples of 2-distinguishable infinite graphs include the denumerable random graph [13], the infinite hypercube of dimension n [13], locally finite trees with no vertex of degree 1 [17], and denumerable vertex-transitive graphs of connectivity 1 [15].

In 2007 Wilfried Imrich posed the following question: "What is the minimum number of vertices in a label class of a 2-distinguishing labeling for the hypercube Q_n?" To aid in addressing this question, call a label class in a 2-distinguishing labeling a *distinguishing class*. Call the minimum size of a distinguishing class for a 2-distinguishable graph G the *cost of* 2-*distinguishing* G and denote it by $\rho(G)$. The 2-distinguishing labeling provided for Q_n by Bogstad and Cowan in

[4] shows that for $n \geq 4$, $\rho(Q_n) \leq n + 2$. The best result known when Imrich asked the question publicly was $\rho(Q_n) \approx \sqrt{n}$ [11]. In [6] this author showed that for $n \geq 5$, $\lceil \log_2 n \rceil + 1 \leq \rho(Q_n) \leq 2\lceil \log_2 n \rceil - 1$.

A significant tool used in finding distinguishing classes is the *determining set* [5], a set of vertices whose point stabilizer is trivial. By definition, a distinguishing class has trivial set stabilizer. Thus it also has trivial pointwise stabilizer and is therefore a determining set. In particular, a set of vertices is a distinguishing class if and only if it is a determining set with the property that any automorphism that fixes it setwise must also fix it pointwise. Intuitively such a set has *no internal symmetries*. Thus the cost of 2-distinguishing a graph G is bounded below by the size of a smallest determining set. This latter size is called the *determining number* of the graph and is denoted $\det(G)$. In [8], this author showed that for almost every graph G that is prime with respect to the Cartesian product (not including K_2), for all but a handful of n, $\rho(G^n) \in \{\det(G^n), \det(G^n) + 1\}$. In this paper we see that for selected Kneser graphs and selected hypercubes $\rho(G) = \det(G) + 1$. The distinguishing class for Kneser graphs is found first, and then is adapted to provide a distinguishing class for hypercubes. These provide the only results on the cost of 2-distinguishing Kneser graphs so far, while providing a significant improvement on cost of 2-distinguishing hypercubes given by this author in [6].

The paper is organized as follows. Definitions and facts about determining sets and distinguishing labelings are given in Section 2. In Section 3 we show that $\rho(K_{2^m - 1 : 2^{m-1} - 1}) = m + 1$, while in Section 4 we show that $\rho(Q_{2^m - 2}) = \rho(Q_{2^m - 1}) = \rho(Q_{2^m}) = m + 2$. Finally, in Section 5 we provide some open problems.

2 Background

2.1 Determining Sets

Let G be a graph. A subset $\mathcal{A} \subseteq V(G)$ is said to be a *determining set* for G if whenever $\varphi, \psi \in \mathrm{Aut}(G)$ so that $\varphi(x) = \psi(x)$ for all $x \in \mathcal{A}$, then $\varphi = \psi$. Thus every automorphism of G is uniquely determined by its action on the vertices of a determining set. The determining set is an example of a *base of a permutation group action*. Every graph has a determining set since a set containing all but one vertex of the graph is determining. Define the *determining number* of G to be the minimum size of a determining set for G and denote it by $\det(G)$.

Recall that the *set stabilizer* of $\mathcal{A} \subseteq V(G)$ is the set of all $\varphi \in \mathrm{Aut}(G)$ for which $\varphi(x) \in \mathcal{A}$ for all $x \in \mathcal{A}$. In this case we say that $\mathcal{A}$ is invariant under φ and we write $\varphi(\mathcal{A}) = \mathcal{A}$. The *point stabilizer* of $\mathcal{A}$ is the set of all $\varphi \in \mathrm{Aut}(G)$ for which $\varphi(x) = x$ for all $x \in \mathcal{A}$. It is easy to see that $\mathcal{A} \subseteq V(G)$ is a determining set for G if and only if the point stabilizer of $\mathcal{A}$ is trivial.

2.2 Distinguishing Labelings

A labeling $f : V(G) \to \{1, \dots, d\}$ is said to be *d-distinguishing* if only the trivial automorphism preserves the label classes. Every graph has a distinguishing labeling since each vertex can be assigned a distinct label. A graph is called *d-distinguishable* if it has a *d*-distinguishing labeling. If G is a 2-distinguishable graph, call a label class in a 2-distinguishing labeling of G a *distinguishing class*. Define the *cost of 2-distinguishing* G to be the minimum size of a distinguishing class for G and denote it by $\rho(G)$.

The following lemma ties together determining sets and distinguishing labelings and facilitates the work in this paper.

Lemma 1. [6] A subset of vertices $\mathcal{A}$ is a distinguishing class for G if and only if $\mathcal{A}$ is a determining set for G with the property that every automorphism that fixes $\mathcal{A}$ setwise, also fixes it pointwise.

In particular, suppose $\mathcal{A}$ is such a determining set. Label the vertices of $\mathcal{A}$ with 1s and the vertices of its complement with 2s. Suppose that $\varphi \in \mathrm{Aut}(G)$ preserves the label classes. Then φ is in the set stabilizer of $\mathcal{A}$, which by hypothesis means that it is in the point stabilizer of $\mathcal{A}$. Since $\mathcal{A}$ is a determining set, this means φ is the identity. Thus $\mathcal{A}$ is a distinguishing class.

3 Selected Kneser Graphs

Recall that vertices in the Kneser graph $K_{n:k}$ are subsets of $[n] = \{1, \dots, n\}$ of size k, with edges between pairs of disjoint subsets. Thus vertices of $K_{n:k}$ will be thought of simultaneously as vertices of $K_{n:k}$ and as subsets of $[n]$. For the purpose of this paper we will assume that $n > 2k$. Recall that Kneser graphs are a generalization of the odd graphs. More particularly, $K_{2^m-1:2^{m-1}-1}$ is the odd graph $O(2^m)$. The notation for odd graphs is simpler, but we will stay with the more general concept and notation of Kneser graphs.

Let $\mathcal{V} = \{V_1, \dots, V_m\}$ be an ordered subset of vertices of $K_{n:k}$. For each $V \in \mathcal{V}$, let the *characteristic vector* of V be the binary n-tuple with a one in position i if and only if $i \in V$. Let the *characteristic matrix* of $\mathcal{V}$, denoted $M_\mathcal{V}$, be a matrix whose j^{th} row is the characteristic vector of V_j. Recall that the automorphism group of $K_{n:k}$ consists of the permutations of $[n]$ applied to the k-subsets of $[n]$. The effect of an automorphism on a characteristic vector (resp. characteristic matrix) is the application of the permutation to the coordinates of the vector (resp. the columns of the matrix).

Let $\varphi \in \mathrm{Aut}(K_{n:k})$ so that $\varphi(\mathcal{V}) = \mathcal{V}$. It will be useful to understand the effect of φ on $M_\mathcal{V}$. Since $\varphi(\mathcal{V}) = \mathcal{V}$, $M_{\varphi(\mathcal{V})}$ has the same rows as $M_\mathcal{V}$ but (potentially) in a different order. That is, there is a permutation $\sigma \in S_m$ on the rows of $M_\mathcal{V}$ that yields $M_{\varphi(\mathcal{V})}$. However, since $\varphi \in \mathrm{Aut}(K_{n:k})$ there is a permutation $\pi \in S_n$ on the columns of $M_\mathcal{V}$ so that the jth column of $M_{\varphi(\mathcal{V})}$ is the $\pi^{-1}(j)^{th}$ column of $M_\mathcal{V}$. Note that φ fixes $\mathcal{V}$ pointwise if and only if $M_\mathcal{V} = M_{\varphi(\mathcal{V})}$.

$$M_\mathcal{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 0101010101010101010101010101010 \\ 1001100110011001100110011001100 \\ 1110000111100001111000011110000 \\ 1111111000000001111111100000000 \\ 1111111111111110000000000000000 \end{bmatrix}$$

Figure 1: The characteristic matrix $M_\mathcal{V}$ for $K_{31:15}$

Our goal is to find a distinguishing class for $K_{2^m-1:2^{m-1}-1}$. By Lemma 1, we need a determining set $\mathcal{V}$ which is fixed pointwise by every automorphism that fixes it setwise. Thus we want that if $\varphi(\mathcal{V}) = \mathcal{V}$ then $M_\mathcal{V} = M_{\varphi(\mathcal{V})}$.

Theorem 1. $\rho(K_{2^m-1:2^{m-1}-1}) = m + 1$.

Proof. This theorem is proved in two parts. First we show that the cost of 2-distinguishing $K_{2^m-1:2^{m-1}-1}$ is greater than m. Next we show that $K_{2^m-1:2^{m-1}-1}$ has a distinguishing class of size $m + 1$.

From [5] we know that a set is a minimum size determining set for $K_{2^m-1:2^{m-1}-1}$ if and only if its characteristic matrix consists of the $2^m - 1$ distinct columns of zeros and ones, excluding the column of all ones. Thus the characteristic matrices of any two minimum size determining sets for $K_{2^m-1:2^{m-1}-1}$ are images of each other under a permutation of their columns. Thus the sets themselves are images of each other under the corresponding automorphism of $K_{2^m-1:2^{m-1}-1}$. In particular, all minimum size determining sets for $K_{2^m-1:2^{m-1}-1}$ have isomorphic set stabilizers. Thus either every minimum size determining set for $K_{2^m-1:2^{m-1}-1}$ is a distinguishing class, or none is.

A minimum size determining set for $K_{2^m-1:2^{m-1}-1}$ is $\mathcal{V} = \{V_1, \ldots, V_m\}$ with the V_i defined as in [5]. Note that since each vertex (as a subset) contains $2^{m-1} - 1$ elements, each characteristic vector contains $2^{m-1} - 1$ ones and 2^{m-1} zeros. The vertex V_1 has characteristic vector that alternates 0s and 1s, beginning with and ending with a 0. The vertex V_2 has characteristic vector that begins with one 1 and then alternates pairs of 0s with pairs of 1s. In general, V_i has characteristic vector with $2^{i-1} - 1$ 1s followed by alternating strings of 2^{i-1} 0s and 2^{i-1} 1s. Let $M_\mathcal{V}$ be the characteristic matrix for $\mathcal{V}$. See Figure 1 for $M_\mathcal{V}$ for $K_{31:15}$.

Note that transposing V_m and V_{m-1} while fixing the remainder of the vertices of $\mathcal{V}$ can be accomplished by the column permutation τ that transposes columns $2^{m-2}, \ldots, 2^{m-1} - 1$ with columns $2^{m-1}, \ldots, 2^{m-1} + 2^{m-2} - 1$ respectively. More formally, let $\tau = (2^{m-2}\ \ 2^{m-1})\,(2^{m-2}+1\ \ 2^{m-1}+1) \cdots (2^{m-1} - 1\ \ 2^{m-1}+2^{m-2}-1)$. The column permutation τ induces $\varphi \in \text{Aut}(K_{2^m-1:2^{m-1}-1})$ so that φ transposes V_m and V_{m-1} and fixes the rest of $\mathcal{V}$. Thus, τ is in the set stabilizer of $\mathcal{V}$. Therefore $\mathcal{V}$, and every other minimum size determining set for $K_{2^m-1":2^{m-1}-1}$, has non-trivial set stabilizer and fails to be a distinguishing class. Thus $\rho(K_{2^m-1:2^{m-1}-1}) > m$. In what follows, we will construct a distinguishing class of size $m + 1$ for $K_{2^m-1:2^{m-1}-1}$.

$$M_{\mathcal{A}} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ Y \end{bmatrix} = \begin{bmatrix} 0101010101010101010101010101010 \\ 1001100110011001100110011001100 \\ 1110000111100001111000011110000 \\ 1111111000000001111111100000000 \\ 1111111111111110000000000000000 \\ 1111111000000000111000011000101 \end{bmatrix}$$

Figure 2: The characteristic matrix $M_{\mathcal{A}}$ for $K_{31:15}$

Notice that each pair of vertices of $\mathcal{V}$ as defined above (as subsets) has an intersection of size $2^{m-2} - 1$. By [16], since $1 \leq n - 2k \leq k - 1$ the distance between two vertices in the Kneser graph is $\min\{2\lceil \frac{k-t}{n-2k}\rceil, 2\lceil \frac{t}{n-2k}\rceil + 1\}$ where t is the size of the set intersection of the two vertices. For $n = 2^m - 1$ and $k = 2^{m-1} - 1$, this distance is $\min\{2^m - 2 - 2t, 2t + 1\}$. Thus for vertices in $\mathcal{V}$, this graph distance is $2^{m-1} - 1$.

We will define a vertex Y in $K_{2^m-1:2^{m-1}-1}$ whose distances to the vertices of $\mathcal{V}$ are distinct. Define Y as a subset of $[2^m - 1]$ as follows: $Y =$

$$[1, 2^{m-2} - 1] \cup_{i=2}^{m-2} [(2^i - 2)2^{m-i}, (2^i - 2)2^{m-i} + 2^{m-i-1} - 1] \cup \{2^m - 3, 2^m - 1\}.$$

Note that $|Y| = (2^{m-2}-1)+2^{m-3}+\cdots+2^2+2^1+2 = 2^{m-2}+\cdots+2^0 = 2^{m-1}-1$. Let $\mathcal{A} = \mathcal{V} \cup \{Y\}$. See Figure 2 for $M_{\mathcal{A}}$ for $K_{31:15}$.

Lemma 2 (whose statement and proof follow the proof of this theorem) shows that each distance $d(Y \cap V_i)$ is distinct. Since the vertices in $\mathcal{V}$ are pairwise equidistant, and the same is not true for any pair involving Y, Y is distinguished in the set $\mathcal{A} = \mathcal{V} \cup \{Y\}$. Further, since each V_i has a distinct distance to Y, each V_i is distinguished in this set by its distance to Y. Thus $\mathcal{A}$ is a determining set with no internal symmetry and therefore is a distinguishing class of size $m + 1$ for $K_{2^m-1:2^{m-1}-1}$. □

Lemma 2. If $i \neq j$ then $d(Y, V_i) \neq d(Y, V_j)$.

Proof. Recall that Y is built in blocks of consecutive integers. The first block has size $2^{m-2} - 1$; the second block has size 2^{m-3}; the third 2^{m-4}. The $(r-2)^{nd}$ block has size 2, while the $(r-1)^{st}$ and the rth have size 1. In particular, for $2 \leq i \leq r - 2$ the ith block has size 2^{m-i-1}, the same as the size of the blocks in V_{r-i}.

Since V_1 contains precisely the even numbers from $[1, 2^m - 1]$, $|Y \cap V_1|$ is the number of even numbers in Y which is $2^{m-2} - 2$. Since V_m contains only the integers $1, \ldots, 2^{m-1} - 1$, the intersection is precisely the first block of Y. Thus $|Y \cap V_m| = 2^{m-2} - 1$. The first block of V_{m-1} matches the first block of Y. This accounts for $2^{m-2} - 1$ elements in their intersection. The second block of V_{m-1} contains precisely the second block of Y - precisely 2^{m-3} elements. Thus $|Y \cap V_{m-1}| = 2^{m-2} + (2^{m-3} - 1)$. The intersection sizes for other V_i can be similarly computed.

From [16] we know if $|Y \cap V_j| = t$ then $d(Y, V_j) = \min\{2^m - 2 - 2t, 2t + 1\}$. This yields the following.

$$
\begin{array}{llll}
|Y \cap V_1| & = & 2^{m-2} - 2, & \qquad d(Y, V_1) & = & 2^{m-1} - 3 \\
|Y \cap V_2| & = & 2^{m-2}, & \qquad d(Y, V_2) & = & 2^{m-1} - 2 \\
|Y \cap V_3| & = & 2^{m-2} + 1, & \qquad d(Y, V_3) & = & 2^{m-1} - 4 \\
|Y \cap V_4| & = & 2^{m-2} + 3, & \qquad d(Y, V_4) & = & 2^{m-1} - 8 \\
& \vdots & & & \vdots & \\
|Y \cap V_{m-3}| & = & 2^{m-2} + 2^{m-5} - 1, & \qquad d(Y, V_{m-3}) & = & 2^{m-1} - 2^{m-4} \\
|Y \cap V_{m-2}| & = & 2^{m-2} + 2^{m-4} - 1, & \qquad d(Y, V_{m-2}) & = & 2^{m-1} - 2^{m-3} \\
|Y \cap V_{m-1}| & = & 2^{m-2} + 2^{m-3} - 1, & \qquad d(Y, V_{m-1}) & = & 2^{m-1} - 2^{m-2} \\
|Y \cap V_m| & = & 2^{m-2} - 1, & \qquad d(Y, Y_m) & = & 2^{m-1}
\end{array}
$$

Thus each $V_i \in \mathcal{V}$ has a distinct distance to Y in $K_{2^m - 1 : 2^{m-1} - 1}$. $\qquad\qquad \square$

4 Selected Hypercubes

Recall that Q_n is the Cartesian product of n copies of K_2. Each automorphism of K_2^n can be understood in the following way.

Theorem 2. [10] If $\varphi \in \mathrm{Aut}(K_2^n)$ there is a permutation $\pi \in S_n$ and isomorphisms $\psi_i : K_2 \to K_2$ so that

$$
\varphi(v_1, \ldots, v_m) = (\psi_{\pi^{-1}(1)}(v_{\pi^{-1}(1)}), \ldots, \psi_{\pi^{-1}(m)}(v_{\pi^{-1}(m)})).
$$

That is, each automorphism of Q_n is a permutation of the K_2 factors, composed with automorphisms of individual factors. That is, $\mathrm{Aut}(K_2^n)$ is the wreath product of $\mathbb{Z}_2^n$ and S_n.

Let $\mathcal{A} = \{V_1, \ldots, V_m\}$ be an ordered set of vertices of Q_n. Let $M_\mathcal{A}$ be the $m \times n$ matrix whose i^{th} row contains the coordinates for V_i with respect to the Cartesian product $Q_n = K_2^n$. (Note that we identify the vertices of K_2 with 0 and 1.) Call $M_\mathcal{A}$ the *characteristic matrix* of $\mathcal{A}$. Note that the jth column of $M_\mathcal{A}$ consists of the jth coordinates of $V_1, \ldots, V_m$ and can be denoted $[V_{1,j} \ \ldots \ V_{m,j}]^T$. We say the jth and k^{th} columns of $M_\mathcal{A}$, $[V_{1,j} \ \ldots \ V_{m,j}]^T$ and $[V_{1,k} \ \ldots \ V_{m,k}]^T$, are *isomorphic images of each other* if there exists an isomorphism $\psi : K_2 \to K_2$ so that $\psi(V_{i,j}) = V_{i,k}$ for all i. We can now state criteria for a set to be a determining for Q_n as follows.

Lemma 3. [7] A set of vertices $\mathcal{A}$ is a determining set for Q_n if and only if no two columns of $M_\mathcal{A}$ are isomorphic images of each other.

Theorem 3. For $m \geq 3$, $\rho(Q_{2^m}) = \rho(Q_{2^m - 1}) = \rho(Q_{2^m - 2}) = m + 2$.

Proof. This theorem is proved in two parts. First we show that the cost of 2-distinguishing each of the given graphs is greater than $m + 1$. Next we show that each of these graphs has a distinguishing class of size $m + 2$.

We know by Corollary 1 that every distinguishing class is a determining set, and we know from [7] that each of the graphs in question has determining number $m + 1$. Thus we need to show for each of these graphs that no determining set of minimum size can be a distinguishing class.

For Q_{2^m}: A canonical minimum size determining set for Q_{2^m} is given in [7]. It is $\mathcal{T} = \{T_0, \ldots, T_m\}$ where T_0 is the 2^m-bit string of all zeros, and for $1 \leq i \leq m$, T_i is the 2^m-bit binary string that alternates 2^{i-1} consecutive ones with 2^{i-1} consecutive zeros. Let $M_\mathcal{T}$ be the characteristic matrix for $\mathcal{T}$. See Figure 3 for $M_\mathcal{T}$ when $m = 5$. Since $\mathcal{T}$ is a determining set, by Lemma 3, no two columns are isomorphic, and therefore no two are equal. Further, the first entry in each column of $M_\mathcal{T}$ is 0. Since there exist precisely 2^m distinct binary columns of length $m+1$ whose first entry is zero, $M_\mathcal{T}$ contains all such columns.

$$M_\mathcal{T} = \begin{bmatrix} 00000000000000000000000000000000 \\ 10101010101010101010101010101010 \\ 11001100110011001100110011001100 \\ 11110000111100001111000011110000 \\ 11111111000000001111111100000000 \\ 11111111111111110000000000000000 \end{bmatrix}$$

Figure 3: The characteristic matrix $M_\mathcal{T}$ for Q_{32}.

Let $\sigma \in S_{m+1}$ be an arbitrary permutation of $\mathcal{T}$. Consider the characteristic matrix $M_{\sigma(T)}$ associated with the ordered set $\sigma(\mathcal{T})$. Since σ changes only the order of the vertices of $\mathcal{T}$, both $\mathcal{T}$ and $\sigma(\mathcal{T})$ are determining sets for Q_{2^m}. Thus no pair of columns of $M_{\sigma(T)}$ are isomorphic images of each other. Apply the trivial or non-trivial automorphisms ψ_i of K_2 to each column of $M_{\sigma(T)}$ so that the resulting matrix N has a 0 in the first position of each column. Since no pair of columns of $M_{\sigma(T)}$ are isomorphic, no pair of columns of N are equal. Thus N has 2^m distinct binary column of length $m + 1$ each of which begins with a zero. Thus the columns of N are the same as the columns of $M_\mathcal{T}$, but permuted by a column permutation, say π. Thus we can get from N to $M_\mathcal{T}$ by π, and therefore from $M_{\sigma(T)}$ to $M_\mathcal{T}$, by π and the ψ_i. By Theorem 2, such a combination of column isomorphisms and permutations corresponds to an automorphism of $K_2^{2^m} = Q_{2^m}$. Thus for any $\sigma \in S_{m+1}$, the permutation σ of the elements of $\mathcal{T}$ can be accomplished by an automorphism of Q_{2^m}. Thus the set stabilizer of $\mathcal{T}$ is isomorphic to S_{m+1}.

Thus the canonical determining set of minimum size for Q_{2^m} has non-trivial set stabilizer. Next we will see that any minimum size determining set for Q_{2^m} is an automorphic image of the canonical one, and therefore also has non-trivial set stabilizer.

Let $\mathcal{U}$ be an arbitrary determining set of minimum size for Q_{2^m} with characteristic matrix N. To each column of N whose first entry is a one, apply the non-trivial automorphism of K_2. Call the result N' and note that each column begins with a zero. Since $\mathcal{U}$ is a determining set, no pair of columns of N are isomorphic images of each other, and thus no pair of columns of N' are equal. Thus N' contains 2^m distinct columns of length $m + 1$ each of which begins with a zero. Since there are precisely 2^m such columns, and these are exactly the columns of $M_\mathcal{T}$, the columns of N' are the columns of $M_\mathcal{T}$ permuted. Thus we can get from N' to N, and therefore from N' to $M_\mathcal{T}$, using column iso-

morphisms and column permutations. But by Theorem 2, such a combination of column isomorphisms and permutations corresponds to an automorphism of $K_2^{2^m} = Q_{2^m}$. Thus, $\mathcal{U}$ is an image of T under an automorphism of Q_{2^m}. Thus $\mathcal{T}$ and $\mathcal{U}$ have the same set stabilizer. Since we have already proved that the set stabilizer of $\mathcal{T}$ is S_{m+1}, $\mathcal{U}$ has non-trivial set stabilizer and thus is not a distinguishing class.

Let $\mathcal{V}$ (resp. $\mathcal{W}$) be a minimum size determining set for Q_{2^m-1} (resp. Q_{2^m-2}). Note that since $m \geq 3$, by [7], $\det(Q_{2^m-1}) = \lceil \log_2(2^m-1) \rceil + 1 = m+1$ (resp. $\det(Q_{2^m-2}) = \lceil \log_2(2^m-2) \rceil + 1 = m+1$). Using the same argument as above, we can apply a non-trivial automorphism of K_2 to each column of $M_{\mathcal{V}}$ (resp. $M_{\mathcal{W}}$) that has a 1 in its first position. Call the resulting matrix $N_{\mathcal{V}}$ (resp. $N_{\mathcal{W}}$). Since $\mathcal{V}$ (resp. $\mathcal{W}$) is a determining set, no two columns of $N_{\mathcal{V}}$ (resp. $N_{\mathcal{W}}$) are equal. Thus $N_{\mathcal{V}}$ (resp. $N_{\mathcal{W}}$) contains 2^m-1 (resp. 2^m-2) distinct columns of length $m+1$ that begin with a 0. Thus there exists a column a (resp. two columns a and b) so that $M_{\mathcal{T}}$ with column a (resp. a and b) deleted has the same columns as $N_{\mathcal{V}}$ (resp. $N_{\mathcal{W}}$). Denote the vertices given by the rows of the matrix $M_{\mathcal{T}}$ with column a (resp. columns a and b) deleted by $\mathcal{T}_a$ (resp. $\mathcal{T}_{ab}$). Thus we can get from $N_{\mathcal{V}}$ (resp. $N_{\mathcal{W}}$) to $M_{\mathcal{T}_a}$ (resp. $M_{\mathcal{T}_{ab}}$) by column permutation, and therefore from $M_{\mathcal{V}}$ (resp. $M_{\mathcal{W}}$) to $M_{\mathcal{T}_a}$ (resp. $M_{\mathcal{T}_{ab}}$) by column isomorphisms and column permutation. Thus $\mathcal{V}$ (resp. $\mathcal{W}$) is an automorphic image of $\mathcal{T}_a$ (resp. $\mathcal{T}_{ab}$). Thus we need only show that $\mathcal{T}_a$ (resp. $\mathcal{T}_{ab}$) is a determining set of minimum size for Q_{2^m-1} (resp. Q_{2^m-2}) with non-trivial set stabilizer. It is easy to see that $\mathcal{T}_a$ (resp. $\mathcal{T}_{ab}$) is a determining set since its columns are still pairwise non-isomorphic.

For Q_{2^m-1}: Let A be the set of vertices in $\mathcal{T}$ that have a one in coordinate a. Let $\sigma \in S_{m+1}$ be a non-trivial permutation of $\mathcal{T}$ under which each of A and A^c is invariant. Such a permutation exists since $m \geq 3$ and therefore one of A and A^c contains more than one element. We have shown that every permutation σ of $\mathcal{T}$ corresponds to a column permutation π with column isomorphisms ψ_i, and thus to an automorphism of Q_{2^m}. By our choice of σ we guarantee that it preserves the value of the ath coordinate of vertex T_i in $\mathcal{T}$. Thus column a of $M_{\mathcal{T}}$ is fixed by π and by ψ_a. Let π' be the permutation induced by π on $[2^m] - \{a\}$. Then π' with the given ψ_i induces σ on $\mathcal{T}_a$. Thus the set stabilizer of $\mathcal{T}_a$ is non-trivial. Thus $\mathcal{T}_a$, and therefore $\mathcal{V}$, is not a distinguishing set for Q_{2^m-1}.

For Q_{2^m-2}: Let A be as above and let B be the set of vertices of $\mathcal{T}$ that have a one in position b. Consider sets $A \cap B$, $A^c \cap B^c$, $A \cap B^c$, and $A^c \cap B$. Note that these sets partition $\mathcal{T}$. Suppose that one of these sets contains at least two vertices, say V_i and V_j. Let $\tau \in S_{m+1}$ be the permutation of $\mathcal{T}$ that transposes V_i and V_j. Then $\mathcal{T}$ is invariant under τ and, in particular, each of A, A^c, B, B^c is also invariant. Thus the positions of the values in columns a and b of $M_{\mathcal{T}}$ and $M_{\tau(T)}$ are the same. But $\tau \in S_{m+1}$ induces a non-trivial automorphism of Q_{2^m} under which $\mathcal{T}$ is invariant and whose effect on $M_{\mathcal{T}}$ is given by column permutation π and isomorphisms ψ_i. Thus we see that the set of columns $\{a, b\}$ are invariant under π.

Suppose that none of $A \cap B, A^c \cap B^c, A \cap B^c, A^c \cap B$ has at least two vertices.

Then $m+1 = 4$ and (up to the order of the rows) columns a and b are $[0\ 0\ 1\ 1]^T$ and $[0\ 1\ 0\ 1]^T$ respectively. Then $V_2 \in A^c \cap B$ and $V_3 \in A \cap B^c$. Let $\tau \in S_4$ be the permutation that transposes V_2 and V_3. Then $M_{\tau(\mathcal{T})}$ has column a equal to column b of $M_{\mathcal{T}}$ and vice versa. But $\tau \in S_{m+1}$ induces a non-trivial automorphism of Q_{2^m} under which $\mathcal{T}$ is invariant and whose effect on $M_{\mathcal{T}}$ is given by column permutation π and isomorphisms ψ_i. Thus we see that the set of columns $\{a, b\}$ is invariant under π.

Thus regardless of the size of the sets $A \cap B, A^c \cap B^c, A \cap B^c, A^c \cap B$, since $\{a, b\}$ is invariant under π, we may let π'' be the permutation induced by π on $[2^m] - \{a, b\}$. Then π'' with the appropriate given ψ_i induces τ on $\mathcal{T}_{ab}$. Thus the set stabilizer of $\mathcal{T}_{ab}$ is non-trivial. Thus $\mathcal{T}_{ab}$, and therefore $\mathcal{W}$, is not a distinguishing set for $Q_{2^{m-1}}$.

Thus $\rho(Q_{2^m}), \rho(Q_{2^m-1}), \rho(Q_{2^m-2}) > m+1$. In what follows we construct a distinguishing class of size $m+2$ for each of Q_{2^m}, Q_{2^m-1}, and Q_{2^m-2}.

For Q_{2^m-1}: Since a characteristic vector of a vertex in $K_{2^m-1:2^{m-1}-1}$ is a binary vector of length $2^m - 1$, we can think of such a characteristic vector as a vertex of Q_{2^m-1}. Let $V_1, \ldots, V_m, Y$ be the vertices of Q_{2^m-1} associated with the characteristic vectors of vertices $V_1, \ldots, V_m, Y$ for the minimum distinguishing class for $K_{2^m-1:2^{m-1}-1}$ given in Section 3. Let V_0 be vertex represented by the sequence of $2^m - 1$ zeros. Let $\mathcal{A} = \{V_0, \ldots, V_m, Y\}$. Since no two columns in the characteristic matrix of $\mathcal{A}$ are isomorphic, $\mathcal{A}$ is a determining set for Q_{2^m-1}.

Though we can think of vertices in $K_{2^m-1:2^{m-1}-1}$ as vertices in Q_{2^m-1}, the graph distances between such vertices are different in the two types of graphs. The distance in $K_{2^m-1:2^{m-1}-1}$ can be computed as in [16] using the size of the intersection of the corresponding subsets. Further, the size of the intersection is the number of positions in which both characteristic vectors have ones. The distance in Q_{2^m-1} is the number of positions in which the vectors differ. Suppose $|Y \cap V_i| = r$ when $Y \cap V_i$ is considered as a subset of the vertices of $K_{2^m-1:2^{m-1}-1}$. Then there are r coordinates in which both Y and V_i have ones. Since each of Y and V_i has precisely $2^m - 1$ ones, this means that there are $2^m - 1 - r$ positions in which Y has ones but V_i has zeros and $2^m - 1 - r$ positions in which V_i has ones but Y has zeros. Thus there are $2(2^m - 1 - r)$ positions in which they differ. This is their distance in the graph Q_{2^m-1}.

Note that in Q_{2^m-1} the V_i are pairwise equidistant, the distance between V_0 and Y is 2^{m-1}, and the distance between Y and V_i is $2(2^m - 1 - |Y \cap V_i|)$. For $1 \le i, j \le m$, whenever $i \ne j$ the fact that $|Y \cap V_i| \ne |Y \cap V_j|$ (as subsets of $K_{2^m-1:2^{m-1}-1}$) tells us that $d(Y, V_i) \ne d(Y, V_j)$ (in Q_{2^m-1}). Further since $2(2^m - 1 - |Y \cap V_i|)$ is even and $2^{m-1} - 1$ is odd, $d(Y, V_0) \ne d(Y, V_i)$ for all $1 \le i \le m$. Thus the distances of $V_0, \ldots, V_m$ to Y are all distinct. Thus the set $\mathcal{A} = \{V_0, \ldots, V_m, Y\}$ has no internal symmetry and is a distinguishing class for Q_{2^m-1}.

For Q_{2^m-2}: Begin with the distinguishing class $\mathcal{A}$ for Q_{2^m-1} and delete the rightmost position from each vertex. Since Y was one in this position and all V_i were zero, this reduces all distances between Y and the V_i by one. Thus all V_i are still distinguished by their distance to Y. This is a determining set since

we have maintained the property that no two columns are isomorphic images of each other. Thus this is a distinguishing class for Q_{2^m-2}.

For Q_{2^m}: Begin with the distinguishing class $\mathcal{A}$ for Q_{2^m-1} and add a one as the leftmost position in each vertex. Since all vertices agree in this position, all distances remain the same. Thus the distances from Y still distinguish this set. Since there is, by construction, no other position of all ones or of all zeros, this remains a determining set. Thus this is a distinguishing class for Q_{2^m}. $\quad\square$

5 Open Problems

Problem 1. Find precise values for the remaining $\rho(Q_n), \rho(K_{n:k})$. These are likely to be larger than the determining number plus one.

Problem 2. Classify graphs for which $\rho(G) = \det(G)$.

Problem 3. Classify graphs for which $\rho(G) = \det(G) + 1$.

Problem 4. Find graphs for which $\rho(G)$ is arbitrarily larger than $\det(G)$.

Problem 5. Find more minimum size determining sets for families of 2-distinguishable graphs to help find the cost of 2-distinguishing.

6 Acknowledgments

References

[1] Michael O. Albertson. Distinguishing Cartesian powers of graphs. *Electron. J. Combin.*, 12:Note 17 (electronic), 2005.

[2] Michael O. Albertson and Debra L. Boutin. Using determining sets to distinguish Kneser graphs. *Electron. J. Combin.*, 14(1):Research Paper 20 (electronic), 2007.

[3] Michael O. Albertson and Karen L. Collins. Symmetry breaking in graphs. *Electron. J. Combin.*, 3(1):Research Paper 18 (electronic), 1996.

[4] Bill Bogstad and Lenore J. Cowen. The distinguishing number of the hypercube. *Discrete Math.*, 283(1-3):29–35, 2004.

[5] Debra L. Boutin. Identifying graph automorphisms using determining sets. *Electron. J. Combin.*, 13(1):Research Paper 78 (electronic), 2006.

[6] Debra L. Boutin. Small label classes in 2-distinguishing labelings. *Ars Math. Contemp.*, 1(2):154–164, 2008.

[7] Debra L. Boutin. The determining number of a Cartesian product. *J. Graph Theory*, 61(2):77–87, 2009.

[8] Debra L. Boutin. The cost of 2-distinguishing Cartesian powers. preprint, 2012.

[9] Tomotaka Fukuda, Seiya Negami, and Thomas W. Tucker. 3-connected planar graphs are 2-distinguishable with few exceptions. *Yokohama Math. J.*, 54(2):143–153, 2008.

[10] Richard Hammack, Wilfried Imrich, and Sandi Klavžar. *Handbook of product graphs*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, second edition, 2011.

[11] Wilfried Imrich. Personal communication.

[12] Wilfried Imrich and Sandi Klavžar. Distinguishing Cartesian powers of graphs. *J. Graph Theory*, 53(3):250–260, 2006.

[13] Wilfried Imrich, Sandi Klavžar, and Vladimir Trofimov. Distinguishing infinite graphs. *Electron. J. Combin.*, 14(1):Research Paper 36, 12 pp. (electronic), 2007.

[14] Sandi Klavžar and Xuding Zhu. Cartesian powers of graphs can be distinguished by two labels. *European J. Combin.*, 28(1):303–310, 2007.

[15] Simon M. Smith, Mark E. Watkins, and Thomas W. Tucker. Distinguishability of infinite groups and graphs. preprint, 2011.

[16] Mario Valencia-Pabon and Juan-Carlos Vera. On the diameter of Kneser graphs. *Discrete Math.*, 305(1-3):383–385, 2005.

[17] Mark E. Watkins and Xiangqian Zhou. Distinguishability of locally finite trees. *Electron. J. Combin.*, 14(1):Research Paper 29, 10 pp. (electronic), 2007.