Distinguishing Generalized Mycielskian Graphs

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Abstract

A graph G is *d-distinguishable* if there is a coloring of the vertices with d colors so that only the trivial automorphism preserves the color classes. The smallest such d is the *distinguishing number*, $\text{Dist}(G)$. The *Mycielskian* $\mu(G)$ of a graph G is constructed by adding a shadow vertex u_i for each vertex v_i of G and one additional vertex w and adding edges so that $N(u_i) = N_G(v_i) \cup \{w\}$. The *generalized Mycielskian* $\mu^{(t)}(G)$ is a Mycielskian graph with t layers of shadow vertices, each with edges to layers above and below. This paper examines the distinguishing number of the traditional and generalized Mycielskian graphs. Notably, if $G \neq K_1, K_2$ and the number of isolated vertices in $\mu^{(t)}(G)$ is at most $\text{Dist}(G)$, then $\text{Dist}(\mu^{(t)}(G)) \leq \text{Dist}(G)$. This result proves and exceeds a conjecture of Alikhani and Soltani.

1 Introduction

Vertex colorings can be a useful way to study the symmetries of a graph, whether or not the automorphism group of the graph is explicitly known. In this paper we study vertex colorings that are not preserved under any nontrivial automorphism. Such colorings are said to be *distinguishing*. The necessary (and sufficient) feature of a distinguishing coloring, and the original reason for its definition, is that every vertex in the graph can be uniquely identified by its graph properties and its color.

More precisely, a coloring of the vertices of a graph G with the colors $1, \ldots, d$ is called a *d-distinguishing coloring* if no nontrivial automorphism of G preserves the color classes. The distinguishing number $\text{Dist}(G)$ of G is the least d such that G has a d-distinguishing coloring. Albertson and Collins introduced graph

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distinguishing in [3]. There has been an increasing amount of interest in graph distinguishing since its introduction.

Most of the work in the last few decades has dealt with large families of graphs and results that show that all but a finite number of graphs in each family have distinguishing number 2. Examples of such families of finite graphs include: hypercubes Q_n with $n \geq 4$ [5], Cartesian powers G^n for a connected graph $G \neq K_2, K_3$ and $n \geq 2$ [1, 7, 9], Kneser graphs $K_{n:k}$ with $n \geq 6, k \geq 2$ [2], and (with seven small exceptions) 3-connected planar graphs [6]. Examples of such families of infinite graphs include: the denumerable random graph [8], the infinite hypercube [8], locally finite trees with no vertex of degree 1 [16], and denumerable vertex-transitive graphs of connectivity 1 [11].

The focus of this paper is on the distinguishing number of graphs achieved by applying the traditional Mycielski construction [10] and the generalized Mycielski construction [12, 13, 14, 15] to simple graphs. Both constructions are formally defined in Section 2. Mycielski's traditional construction on a graph produces one with a strictly larger chromatic number. This construction preserves the property of being triangle-free and Mycielski used it to prove that there exist triangle-free graphs with arbitrarily large chromatic number. The generalized Mycielskian, introduced by Stiebitz [12] in 1985 (cited in [13]) and independently by Van Ngoc [14] in 1987 (cited in [15]), has a similar use. It is defined so the resulting graph has no small odd cycles and, for particular graph inputs, arbitrarily large chromatic number.

In this paper, all graphs are finite simple graphs. We will denote the number of vertices of G by $|G|$, the degree of a vertex v by $d(v)$, and its set of neighbors by $N(v)$. Two vertices x and y are called *twins* if $N(x) = N(y)$. A graph having no twins is said to be *twin-free*. For example, vertices v_1, v_2, and v_3 in Figure 1 are mutually twin vertices; so are u_1, u_2, and u_3. If two vertices of a graph G are twins, then there is an automorphism of G that simply exchanges them and fixes the remaining vertices. Thus, a distinguishing coloring must give distinct colors to each vertex in a set of mutual twins.

Letting $\mu(G)$ denote the (traditional) Mycielskian of a graph G, Alikhani and Soltani [4] proved in 2018 that if G has at least two vertices and is twin-free, then $\mathrm{Dist}(\mu(G)) \leq \mathrm{Dist}(G) + 1$. They then conjectured the following.

Conjecture 1. *[4] Let G be a connected graph of order $n \geq 3$. Then* $\mathrm{Dist}(\mu(G)) \leq \mathrm{Dist}(G)$ *except for a finite number of graphs.*

In Theorem 1 (Section 4), we prove a statement that is slightly stronger than the above conjecture. In particular, we show the conjecture is true for all graphs on at least 3 vertices, not only connected graphs. We also extend our results to generalized Mycielskian graphs. Letting $\mu^{(t)}(G)$ denote the generalized Mycielskian of G with t levels, we prove $\mathrm{Dist}(\mu^{(t)}(G)) \leq \mathrm{Dist}(G)$, unless $G = K_1$, $G = K_2$ and $t = 1$, or the number of isolates in $\mu^{(t)}(G)$ exceeds $\mathrm{Dist}(G)$. In the last case, $\mathrm{Dist}(\mu^{(t)}(G))$ is exactly the number of isolated vertices.

The paper is organized as follows. The definition of the Mycielskian of a graph G, and lemmas regarding automorphisms of $\mu(G)$, are covered in Section 2. The same topics for the generalized Mycielskian of G are developed in Section 3. Theorem 1 on the distinguishing number of $\mu(G)$ and $\mu^{(t)}(G)$ is stated and proved in Section 4.

2 Mycielskian Graphs

In this section, we define and examine the traditional Mycielski construction. Suppose G is a graph with $V(G) = \{v_1, \ldots, v_n\}$. The *Mycielskian of G*, denoted $\mu(G)$, has vertices $\{v_1, \ldots, v_n, u_1, \ldots, u_n, w\}$. For each edge v_iv_j in G, the graph $\mu(G)$ has edges v_iv_j, v_iu_j, and u_iv_j. In addition, $\mu(G)$ has edges u_iw for $i \in \{1, \ldots, n\}$. Thus, $\mu(G)$ has an isomorphic copy of G on vertices $\{v_1, \ldots, v_n\}$. We refer to vertices from $\{u_1, \ldots, u_n\}$ as *shadow vertices* and vertices from $\{v_1, \ldots, v_n\}$ as *original vertices*. Since w dominates the shadow vertices, we refer to w as the *shadow master*.

As an example, $\mu(K_{1,3})$ is shown in in Figure 1.

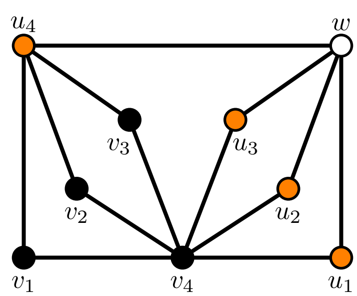

Figure 1: The graph $\mu(K_{1,3})$. The vertices labeled v_i are from $K_{1,3}$, the vertices labeled u_i are the shadow vertices, and w is the shadow master.

We will employ the following properties of $\mu(G)$ and automorphisms throughout our proofs.

Facts about $\mu(G)$: Let $|G| = n$ and $d_G(v_i) = k$. With the notation given above, the Mycielski construction gives us the following: $|\mu(G)| = 2n + 1$; $d_{\mu(G)}(w) = n$; $d_{\mu(G)}(v_i) = 2k$; $d_{\mu(G)}(u_i) = k + 1$; $N_{\mu(G)}(u_i) \setminus \{w\} = N_G(v_i)$; $N_{\mu(G)}(w)$ is an independent set (consisting of all shadow vertices).

For the remainder of this paper, when its use is unambiguous, we will drop the subscript $\mu(G)$ from neighborhoods and degrees. That is, unless otherwise noted, for all $x \in V(\mu(G))$, $N(x) = N_{\mu(G)}(x)$ and $d(x) = d_{\mu(G)}(x)$.

Facts about Automorphisms of G: Let ϕ be an automorphism of a graph G and let $x, y \in V(G)$. Since automorphisms preserve adjacency and nonadjacency of vertex pairs, every property involving adjacency or nonadjacency is also preserved. In particular, degrees: $d(x) = d(\phi(x))$; distances: $d(x, y) = d(\phi(x), \phi(y))$; neighborhoods: $N(x) = N(\phi(x))$.

First we prove that if there is an automorphism of $\mu(G)$ such that the image of w is an original vertex, then G has no dominating vertex.

Lemma 1. *Let G be a graph with $|G| \geq 3$ and let ϕ be an automorphism of $\mu(G)$. If $\phi(w)$ is an original vertex, then G cannot have a dominating vertex.*

Proof. Let $|G| = n$ and assume $\phi(w) = v$ with $d_G(v) = k$. Using facts about $\mu(G)$ and automorphisms we have $d(w) = n$ and so $d(\phi(w)) = d(v) = n$. By construction, $d(v) = 2d_G(v) = 2k$, we have $n = 2k$.

Since $n \geq 3$ and $n = 2k$, we get $k \geq 2$. So $d_G(v) = k = \frac{n}{2} < n-1$ and thus v is not dominating in G. Thus, any dominating vertex of G must be in $N(v)$. However, as $N(w)$ is independent, so is $N(\phi(w)) = N(v)$. Since $d(v) = k \geq 2$, we conclude G has no dominating vertex in $N(v)$, nor thus in G. □

We now show that, in fact, any automorphism of $\mu(G)$ that does not fix the shadow master w must map it to a shadow vertex.

Lemma 2. *Let G be a graph with $|G| \geq 3$. Then no automorphism of $\mu(G)$ maps the shadow master w to any original vertex.*

Proof. Let G be a graph with $n \geq 3$ vertices and suppose by way of contradiction that G has an automorphism ϕ with $\phi(w) = v$ for some original vertex v. We will show that there is no possible image for the shadow of v under ϕ.

Label the vertices of G so that $v = v_n$ and the neighbors of v in G are $\{v_1, \ldots, v_k\}$ with $k < n$. The shadow vertex of v will then be denoted $u = u_n$.

Since u is a shadow vertex, it is adjacent to w by construction, and so $\phi(u)$ is adjacent to $\phi(w) = v$. Thus, $\phi(u) \in N(v) = \{v_1, \ldots, v_k, u_1, \ldots, u_k\}$. We consider two cases: $\phi(u) = u_i$ for some $1 \leq i \leq k$ or $\phi(u) = v_i$ for some $1 \leq i \leq k$ and find a contradiction in each.

Case (I): Suppose that $\phi(u) = u_i$ for some $1 \leq i \leq k$.

We will show that this implies G has a dominating vertex, contradicting Lemma 1.

Since $d(w) = n$ and automorphisms preserve degree, $d(\phi(w)) = d(v) = n$ as well. By construction of $\mu(G)$, we have $d(v) = 2d_G(v) = 2k$. Thus, $n = 2k$.

Since $d_G(v) = k$, by construction $d(u) = k+1$. Further, since automorphisms preserve degree, $d(\phi(u)) = d(u_i) = k + 1$ as well. Since u_i is the shadow vertex of v_i, by construction we also get $d(v_i) = 2k$. Also, by our choice of i, $v_i \in N(v)$. Thus, by properties of the automorphism ϕ^{-1}, we have $\phi^{-1}(v_i) \in N(\phi^{-1}(v)) = N(w)$. Hence, w has a neighbor of degree $2k$.

Since the only neighbors of the shadow master are shadow vertices, there is some j such that $d(u_j) = 2k$. By construction, this means that $d(v_j) = 2(2k-1) = 2n-2$ and so $d_G(v_j) = n-1$. This implies v_j is dominating in G, contradicting Lemma 1. Thus, $\phi(u) \neq u_i$ for any $1 \leq i \leq k$.

Case (II): Suppose that $\phi(u) = v_i$ for some $1 \leq i \leq k$.

We will show that $\phi(v) = w$ and use this to argue that $d(u) = 2$, a contradiction since $d(u) = k + 1$ and $k \geq 2$.

Since $N(u) = \{v_1, \ldots, v_k, w\}$, we have that

$$N(\phi(u)) = N(v_i) = \{\phi(v_1), \ldots, \phi(v_k), \phi(w) = v\}.$$

Since v_i is an original vertex, its neighbors come in original-shadow vertex pairs. In particular, since v is neighbor of v_i, its shadow u must also be a neighbor of v_i, which implies that $u \in \{\phi(v_1), \dots, \phi(v_k)\} \subset N(\phi(v))$. If $u \in N(\phi(v))$, then reciprocally, $\phi(v) \in N(u) = \{v_1, \dots, v_k, w\}$. However, since v is not adjacent to w, $\phi(v)$ is not adjacent to $\phi(w) = v$, which implies $\phi(v) \notin \{v_1, \dots, v_k\}$. Thus $\phi(v) = w$.

Recall that $N(u) \setminus \{w\} = \{v_1, \dots, v_k\}$ is a set of k vertices all adjacent to v. By the properties of automorphisms, it follows that $N(\phi(u)) \setminus \{\phi(w)\} = N(v_i) \setminus \{v\}$ is a set of k vertices all adjacent to $\phi(v) = w$. Therefore $N(v_i) \setminus \{v\}$ must consist entirely of shadow vertices.

Now, by construction, $N(v_i)$ is equally split between original vertices and their corresponding shadow vertices. Since v is the only original vertex in $N(v_i)$, we can conclude that $N(v_i) = \{u, v\}$, so $d(v_i) = 2$. Since $\phi(u) = v_i$ by assumption, $d(u) = 2$ as well. This gives our desired contradiction. □

Lemma 2 leaves only two possibilities for automorphisms that do not fix the shadow master. One is that $|G| < 3$. For example, $\mu(K_2) = C_5$, which is vertex-transitive.

The other way an automorphism might not fix w is to map it to a shadow vertex. For example, Figure 1 shows $\mu(K_{1,3})$ with original vertices in black, shadow vertices in orange, and the shadow master in white. The vertical reflectional symmetry of this drawing induces an automorphism that moves the shadow master to a shadow vertex. Such an automorphism exists for every star graph $K_{1,m}$ with $m \geq 0$. We show in Lemma 3 that star graphs are the only graphs in which the shadow master is not fixed by every automorphism of $\mu(G)$.

Before our next lemma, we introduce the following definition and notation.

Definition. Given a vertex v in a graph, let the *neighborhood degree multiset of* v, denoted D_v, be $\{d(u) : u \in N(v)\}$.

Properties of automorphisms guarantee for every vertex v and automorphism ϕ, that $D_v = D_{\phi(v)}$. We use this fact in the proof of Lemma 3 and in the proofs in Section 3.

Lemma 3. *If there is an automorphism ϕ of $\mu(G)$ that takes the shadow master w to a shadow vertex, then $G = K_{1,m}$ for some $m \geq 0$. Additionally, if $|G| \neq 2$, then $\phi(w)$ is the shadow vertex of the unique vertex of maximum degree in G.*

Proof. Let ϕ be an automorphism of $\mu(G)$ such that $\phi(w)$ is a shadow vertex. Let $|G| = n$ and label the vertices of $\mu(G)$ so that $\phi(w) = u_n$.

If $n = 1$, then $G = K_{1,0}$, and $\mu(G)$ has independent vertex v_1 together with a K_2 consisting of shadow vertex u_1 and shadow master w. Clearly $\phi(w)$ must be u_1, the only other nonisolated vertex in $\mu(G)$.

Suppose $n > 1$. Since $\phi(w) = u_n$, by properties of automorphisms, $D_w = D_{\phi(w)} = D_{u_n}$. We show this equality guarantees $G = K_{1,n-1}$.

By construction of the Mycielskian, we have $N(w) = \{u_1, \dots, u_n\}$ and $d(u_i) = d_G(v_i){+}1$. Thus

$$D_w = \{d_G(v_1){+}1, \dots, d_G(v_n){+}1\}.$$

By construction and properties of graph automorphisms $n = d(w) = d(u_n)$. Then, since u_n is not adjacent to v_n, it must be that $N(u_n) = \{v_1, \dots, v_{n-1}, w\}$. Since $d(v_i) = 2d_G(v_i)$, we see that

$$D_{u_n} = \{2d_G(v_1), \dots, 2d_G(v_{n-1}), d(w)\}.$$

With $D_w = D_{u_n}$ we have

$$\{d_G(v_1)+1, \dots, d_G(v_n)+1\} = \{2d_G(v_1), \dots, 2d_G(v_{n-1}), d(w)\}.$$

Recall $d(w) = n = d(u_n)$ and by construction $d(u_n) = d_G(v_n) + 1$, so removing $d(w) = d_G(v_n) + 1$ yields

$$\{d_G(v_1)+1, \dots, d_G(v_{n-1})+1\} = \{2d_G(v_1), \dots, 2d_G(v_{n-1})\}. \tag{1}$$

We will now show this is impossible when $G \neq K_{1,m}$ for $m \geq 1$. We have already that $d_G(v_n) = n-1$, so suppose that for some value of i with $1 \leq i \leq n-1$, we have $d_G(v_i) > 1$. Define

$$d_{\min} = \min_{1 \leq i \leq n-1} \{d_G(v_i) : d_G(v_i) > 1\}.$$

Let $j \in \{1, \dots, n-1\}$ be such that $d_G(v_j) = d_{\min} > 1$.

Then in Equation 1 on the left hand side $d_G(v_j) + 1$ is the smallest value greater than 2, and on the right hand side, $2d_G(v_j)$ is the smallest value greater than 2. Thus $d_G(v_j) + 1 = 2d_G(v_j)$. However, this can only hold if $d_G(v_j) = 1$, a contradiction of $d_G(v_j) > 1$.

Therefore, we must have $d_G(v_i) = 1$ for $1 \leq i \leq n-1$ and $d_G(v_n) = n-1$. Thus, $G = K_{1,n-1}$ for some $n \geq 2$. Furthermore, if $|G| \geq 3$ then v_n is the unique vertex of maximum degree in G, and $\phi(w)$ is its shadow. □

3 Generalized Mycielskian Graphs

In this section, we define and examine generalized Mycielskian graphs and their automorphisms. The organizational structure and results mirror those in Section 2, although the proofs have some differences.

The generalized Mycielskian of G, also known as a cone over G, was introduced by Stiebitz [12] in 1985 (cited in [13]) and independently by Van Ngoc [14] in 1987 (cited in [15]). For a fixed $t \in \mathbb{N}$ and graph G with vertices $\{v_1, \dots, v_n\}$, the *generalized Mycielskian of* G, written $\mu^{(t)}(G)$, has vertices

$$\{u_1^0, \dots, u_n^0, u_1^1, \dots, u_n^1, \dots, u_1^t, \dots, u_n^t, w\}.$$

For each edge v_iv_j in G, the graph $\mu^{(t)}(G)$ has edges $u_i^0u_j^0$ and u_i^s, u_j^{s+1}, u_j^s, u_i^{s+1}, for $0 \leq s \leq t-1$. In addition, $\mu^{(t)}(G)$ has edges u_i^tw for $1 \leq i \leq n$. Thus, $\mu^{(t)}(G)$ has an isomorphic copy of G on vertices $\{u_1^0, \dots, u_n^0\}$, so we say $u_i^0 = v_i$ for $1 \leq i \leq n$. We say that vertex u_i^s is at level s; the vertices at level 0 are called *original vertices*, and the vertices at level $s \geq 1$ are called *shadow* *vertices (at level s)*. The vertex w is still referred to as the shadow master, though w is only adjacent to the shadow vertices at level t.

In Figure 2, we illustrate both the traditional Mycielskian ($t = 1$) and generalized Mycielskian with $t = 2$, for each of K_2 and K_3. Since $\mu^{(1)}(G) = \mu(G)$, when $t = 1$ we drop the superscript for ease of notation. As before, when subscripts are omitted in degree or neighborhood notation, we are referring to degree or neighborhood in $\mu^{(t)}(G)$.

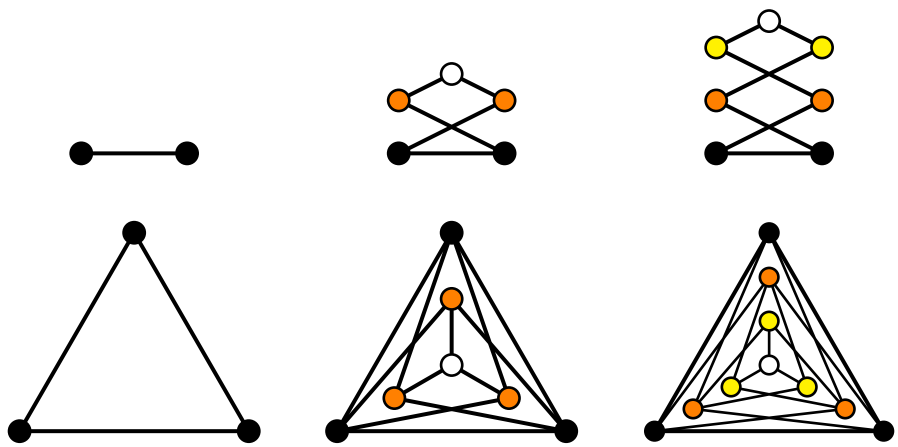

Figure 2: Top: K_2, $\mu(K_2)$ and $\mu^{(2)}(K_2)$, drawn with vertical levels with the shadow master on the top. Bottom: K_3, $\mu(K_3)$ and $\mu^{(2)}(K_3)$, drawn with concentric levels with the shadow master in the middle.

Facts about $\boldsymbol{\mu^{(t)}(G)}$: Let $|G| = n$, $t \in \mathbb{N}$, and $d_G(v_i) = k$. The generalized Mycielski construction gives us the following: $|\mu^{(t)}(G)| = (t+1)n+1$; $d(w) = n$; for $0 \le s \le t-1$, $d(u_i^s) = 2k$; $d(u_i^t) = k+1$; for $1 \le s \le t$, the set of shadow vertices at level s is independent.

The results in Section 2 for the traditional Mycielskian of a graph correspond closely to many of the results for the generalized Mycielskian. To indicate as much, we have labeled appropriate extended results in the same manner as in Section 2, only with a prime added. The exception is Lemma 4, which is only needed for the generalized Mycielskian. As in the case for $\mu(G)$, to prove results about automorphisms of $\mu^{(t)}(G)$, we consider cases based on the image of the shadow master. The following lemma shows that if G is disconnected, then every automorphism of $\mu^{(t)}(G)$ fixes the shadow master.

Lemma 4. *If G is a disconnected graph and ϕ is an automorphism of $\mu^{(t)}(G)$, then ϕ maps the shadow master w to itself.*

Proof. We show here that under the given hypotheses, w is the only vertex of $\mu^{(t)}(G)$ whose removal increases the number of connected components. That is, w is the only cut-vertex in $\mu^{(t)}(G)$. Since every graph automorphism must preserve properties of connectedness, every automorphism of $\mu^{(t)}(G)$ must, therefore, map w to itself.

First, consider the deletion of w. Let v_i and v_j be in distinct components of G. By the Mycielski construction, u_i^t and u_j^t are both adjacent to w in $\mu^{(t)}(G)$. However, u_i^t, w, u_j^t is the only path between u_i^t and u_j^t and so in $\mu^{(t)}(G) \setminus \{w\}$, we have u_i^t and u_j^t in distinct components. This shows that $\mu^{(t)}(G) \setminus \{w\}$ has more components than $\mu^{(t)}(G)$ and so w is a cut-vertex in $\mu^{(t)}(G)$.

We now consider the deletion of other vertices in $\mu^{(t)}(G)$, all of the form u_i^s for $0 \leq s \leq t$, such that $u_i^0 = v_i$ is either an isolated or a nonisolated vertex in G. We show their deletion from $\mu^{(t)}(G)$ does not increase the number of components.

Consider the vertex u_i^s for $0 \leq s \leq t$ such that u_i^0 is a nonisolated vertex in G. For each neighbor v_j of v_i in G, the following cycle exists in $\mu^{(t)}(G)$: $v_i, u_j^1, u_i^2, \ldots, w, \ldots, u_j^2, u_i^1, v_j, v_i$. Observe that this cycle contains u_i^s and, further, that every neighbor of u_i^s is contained in a cycle of this form. Thus, removing u_i^s from $\mu^{(t)}(G)$ does not disconnect the graph. Hence, $\mu^{(t)}(G) \setminus \{u_i^s\}$ has the same number of components as $\mu^{(t)}(G)$ and u_i^s is not a cut-vertex.

Finally, consider the vertex u_i^s for $0 \leq s \leq t$ such that u_i^0 is an isolated vertex in G. If $s \neq t$, then u_i^s is isolated in $\mu^{(t)}(G)$ and so u_i^s cannot be a cut-vertex. If $s = t$, then u_i^t has w as its only neighbor and is also not a cut-vertex.

It follows that w is the only cut-vertex in $\mu^{(t)}(G)$ and so every automorphism of $\mu^{(t)}(G)$ must fix w. □

Knowing that any automorphism of a disconnected graph fixes the shadow master allows us in many cases to only consider connected graphs G. The following lemma also provides us with a useful structural property. In particular, if G is a graph with at least three vertices and $\mu^{(t)}(G)$ has an automorphism mapping w to an original vertex or a shadow vertex not at level t, then G does not have a dominating vertex.

Lemma 1′. *Let G be a graph with $|G| \geq 3$ and $t \in \mathbb{N}$. Let ϕ be an automorphism of $\mu^{(t)}(G)$. If $\phi(w)$ is a vertex at level s for $0 \leq s \leq t-1$, then G does not have a dominating vertex.*

Proof. Let $|G| = n \geq 3$. Assume that ϕ is an automorphism of $\mu^{(t)}(G)$ with $\phi(w)$ either an original vertex or a shadow vertex at level s for some $1 \leq s \leq t-1$. Label the vertices of G so that $\phi(w) = u_n^s$ and so that $N_G(v_n) = \{v_1, \ldots, v_k\}$, where $k = d_G(v_n)$. If $s = 0$, then $u_n^s = v_n$.

By properties of automorphisms and the generalized Mycielskian construction, $d(\phi(w)) = d(u_n^s) = 2k$ and $d(\phi(w)) = d(w) = n$. Thus, $n = 2k$. With $n \geq 3$ it follows that $k \geq 2$. Since $d_G(v_n) = k = \frac{d(u_n^s)}{2}$ by construction, $d_G(v_n) = \frac{n}{2}$. Further, since $k \geq 2$ and $n \geq 3$, we get $\frac{n}{2} \neq n-1$, so v_n is not a dominating vertex in G. It follows that any dominating vertex in G must be in $N(v_n)$.

Suppose there exists $j \in \{1, \ldots, k\}$ so that v_j is a dominating vertex of G. Then $d_G(v_j) = n-1$, so $d(v_j) = 2(n-1)$. If $s \geq 1$, by construction $d(u_j^{s-1}) =$

$2(n-1)$. Also, since $v_j \in N(v_n)$, if $s \geq 1$, we have $u_j^{s-1} \in N(u_n^s)$, and if $s = 0$, we have $v_j \in N(u_n^0)$. Thus, for any $0 \leq s \leq t-1$, we have a degree $2n-2$ vertex adjacent to $\phi(w) = u_n^s$. By properties of automorphisms, this implies that w has a neighbor of degree $2n-2$. However, by construction, all neighbors of w have degree at most n. Since $n < 2n-2$ for $n \geq 3$, we achieve a contradiction.

Hence, if $\phi(w)$ is a vertex at level s for some $0 \leq s \leq t-1$, then G does not have a dominating vertex. □

We will now show that for $|G| \geq 3$, any automorphism of $\mu^{(t)}(G)$ that does not fix the shadow master w, must map w to a shadow vertex at level t. Note that Lemma 2 addresses the case that $t = 1$.

Lemma 2′. *Let G be a graph with $|G| = n \geq 3$ and $t > 1$. Then no automorphism of $\mu^{(t)}(G)$ maps the shadow master w to u_i^s, for any $1 \leq i \leq n$, $0 \leq s \leq t-1$.*

Proof. Let G be a graph with $|G| = n \geq 3$. By Lemma 4, if G is disconnected, then every automorphism ϕ of $\mu^{(t)}(G)$ satisfies $\phi(w) = w$. Thus, we need only consider the case when G is connected.

Suppose there is an automorphism ϕ of $\mu^{(t)}(G)$ that maps the shadow master w to u_i^s for some $1 \leq i \leq n$, $0 \leq s \leq t-1$. Label the vertices so that $\phi(w) = u_n^s$ and $N_G(v_n) = \{v_1, \ldots, v_k\}$, meaning $d_G(v_n) = k$.

We split the remainder of the proof into cases: $s = t-1$ and $0 \leq s \leq t-2$.

Case (I): Suppose that $\phi(w) = u_n^{t-1}$. By construction and since automorphisms preserve degrees, $2k = d(u_n^{t-1}) = d(w) = n$. We will show that there is no possible image for u_n^{t-1} under ϕ.

Since u_n^{t-1} is distance 2 from w, by properties of automorphisms, $\phi(u_n^{t-1})$ is distance two from $\phi(w) = u_n^{t-1}$. To see the choices for $\phi(u_n^{t-1})$, we need only look at the endpoints of paths of length two from $\phi(w) = u_n^{t-1}$. Recall that shadow vertices at levels $s \in \{1, \ldots, t\}$ are independent sets. Thus, unless $t = 2$, a path of length 2 from u_n^{t-1} must change levels at each vertex. Thus such paths can only take one of the following forms: $u_n^{t-1}u_i^t w$, $u_n^{t-1}u_i^t u_j^{t-1}$, $u_n^{t-1}u_i^{t-2}u_j^{t-1}$, $u_n^{t-1}u_i^{t-2}u_j^{t-3}$, or $u_n^1 v_i v_j$, where the latter two paths require $t \geq 3$ and $t = 2$, respectively. Thus, we consider three subcases: Case (Ia): $\phi(u_n^{t-1}) = w$; Case (Ib): $\phi(u_n^{t-1}) = u_j^{t-1}$ for $1 \leq j \leq n-1$; Case (Ic): either $t \geq 3$ and $\phi(u_n^{t-1}) = u_j^{t-3}$ or $t = 2$ and $\phi(u_n^{t-1}) = v_j$ for $1 \leq j \leq n$.

Case (Ia): Suppose that $\phi(u_n^{t-1}) = w$. First, suppose that $t \geq 3$. Since u_n^{t-3} has distance 2 from u_n^{t-1}, and since automorphisms preserve distances, we must have that $\phi(u_n^{t-3})$ has distance 2 from $\phi(u_n^{t-1}) = w$. Thus, since $\phi(w) = u_n^{t-1}$, we have $\phi(u_n^{t-3}) = u_j^{t-1}$, for some $1 \leq j \leq n-1$. We will show that in fact $\phi(u_n^{t-3}) = u_j^{t-1}$ for some $1 \leq j \leq k$. We then will show this implies that G has a dominating vertex, contradicting Lemma 1′. If $t = 2$, then u_n^0 is distance 2 from u_n^{t-1}, and the following argument still holds, replacing u_n^{t-3} with u_n^0.

First, since $N_G(v_n) = \{v_1, \ldots, v_n\}$, by construction the common neighbors of u_n^{t-1} and u_n^{t-3} are $u_1^{t-2}, \ldots, u_k^{t-2}$.

Thus, these k vertices must be mapped to common neighbors of $\phi(u_n^{t-1}) = w$ and $\phi(u_n^{t-3}) = u_j^{t-1}$. However, the common neighbors of w and u_j^{t-1} are the neighbors of u_j^{t-1} at level t. Thus, u_j^{t-1} has exactly k neighbors on (at) level t. Since these are disjoint from the neighbors of u_n^{t-1} at level t and $n = 2k$, they must be $\{u_{k+1}^t, \ldots, u_n^t\}$. Hence, by construction, $N_G(v_j) = \{v_{k+1}, \ldots, v_n\}$. In particular, $v_j \in N_G(v_n)$ and so $1 \le j \le k$.

We have already shown that $d(u_j^{t-1}) = 2k$, so $d(u_j^{t-2}) = 2k$ as well. Further, since $v_j \in N(v_n)$ we see that $u_j^{t-2} \in N(u_n^{t-1}) = N(\phi(w))$. Therefore, by properties of automorphisms, w also has a neighbor of degree $2k$, say u_i^t. By construction this implies $d_G(v_i) = 2k-1 = n-1$, so v_i is dominating vertex in G. This contradicts Lemma $1'$.

Thus $\phi(u_n^{t-1}) \neq w$.

Case (Ib): Suppose that $\phi(u_n^{t-1}) = u_j^{t-1}$, for some $1 \le j \le n-1$. Note, that since $\phi(w) = u_n^{t-1}$, we cannot have $j = n$. We will show that $d_G(v_n) = k$ must be both even and odd, a contradiction. First we will show that $\phi(u_n^t) = u_i^{t-2}$, for some $1 \le i \le k$.

Since u_n^t is adjacent to w, $\phi(u_n^t)$ is adjacent to $\phi(w) = u_n^{t-1}$. Hence, $\phi(u_n^t)$ must be a neighbor of u_n^{t-1} at level t or at level $t-2$. So, $\phi(u_n^t) \in \{u_1^t, \ldots, u_k^t, u_1^{t-2}, \ldots u_k^{t-2}\}$. We next show that $\phi(u_n^t) = u_i^t$ for any $1 \le i \le k$ leads to a contradiction so that $\phi(u_n^t) = u_i^{t-2}$ for some $1 \le i \le k$.

By construction $d(u_n^t) = k+1$, so as automorphisms preserve degrees, if $\phi(u_n^t) = u_i^t$, then $d(u_i^t) = k+1$. Then, by construction, $d(u_i^{t-2}) = 2k$. Moreover, with $1 \le i \le k$, we have $u_i^{t-2} \in N(u_n^{t-1}) = N(\phi(w))$. So w must also be adjacent to a vertex of degree $2k$, say u_j^t. By construction, since u_j^t is a top-level shadow vertex, $d(u_j^t) = d_G(v_j) + 1$, so $d_G(v_j) = 2k-1 = n-1$. Thus, v_j is a dominating vertex in G contradicting Lemma $1'$. Hence, $\phi(u_n^t) \neq u_i^t$ for any $1 \le i \le k$ and therefore, $\phi(u_n^t) = u_i^{t-2}$ for some $1 \le i \le k$.

Then, since $d_G(v_n) = k$, we have $k+1 = d(u_n^t) = d(\phi(u_n^t)) = d(u_i^{t-2})$. However, since u_i^{t-2} is not a top-level shadow vertex, by construction we also have that $d(u_i^{t-2}) = 2d_G(v_i)$. With $k+1 = 2d_G(v_i)$, k must be odd.

Since u_n^{t-1} and w have k common neighbors, namely $u_1^t, \ldots, u_k^t$, we see that $\phi(u_n^{t-1}) = u_j^{t-1}$ and $\phi(w) = u_n^{t-1}$ must have k common neighbors as well. Since u_j^{t-1} and u_n^{t-1} are at the same level, by construction, their common neighbors must be split evenly between vertices at level t and vertices at level $t-2$. This implies that k is even, a contradiction with our earlier conclusion that k is odd.

Thus $\phi(u_n^{t-1}) \neq u_j^{t-1}$ for any $1 \le j \le n-1$.

Case (Ic): Suppose that either $t \ge 3$ and $\phi(u_n^{t-1}) = u_j^{t-3}$ or $t = 2$ and $\phi(u_n^{t-1}) = v_j = u_j^0$ for some $1 \le j \le n$. Say $\phi(u_n^{t-1}) = u_j^r$ with $r \in \{t-3, 0\}$. Note that if $t = 3$, then $r = 0 = t-3$.

Since automorphisms preserve degrees, the neighborhood degree multisets $D_{u_n^{t-1}}$ and $D_{u_j^r}$ are equal. This will yield a contradiction similar to the one in Lemma 3.

By construction half the neighbors of u_n^{t-1} are at level t with degree $d_G(v_i)+1$ for each $1 \le i \le k$ and half are at level $t-2$ with degree $2d_G(v_i)$ for each

$1 \leq i \leq k$. Thus the neighborhood degree multiset of u_n^{t-1} is

$$D_{u_n^{t-1}} = \{d_G(v_1){+}1, \ldots, d_G(v_k){+}1, 2d_G(v_1), \ldots, 2d_G(v_k)\}. \tag{2}$$

By construction, if $t \geq 4$, then $r = t{-}3 > 0$ and so a vertex at level r has half its neighbors at level $t-4$ and the other half at level $t-2$. If $t = 2$ or $t = 3$, then $r = 0$ and so a vertex at level r has half its neighbors at level 0 and the other half at level 1. Thus the neighbors of $\phi(u_n^{t-1}) = u_j^r$ are not at level t, and therefore have degree $2d_G(v_i)$ for some $1 \leq i \leq n$.

To be more precise about $N(u_j^r)$, let $d_G(v_j) = \ell$ and write $N_G(v_j) = \{v_{i_1}, \ldots, v_{i_\ell}\}$ for appropriate indices i_j. By construction, if $t \geq 4$, we have $N(u_j^r) = N(u_j^{t-3}) = \{u_{i_1}^{t-4}, \ldots, u_{i_\ell}^{t-4}, u_{i_1}^{t-2}, \ldots, u_{i_\ell}^{t-2}\}$ and therefore

$$D_{u_j^r} = \{2d_G(v_{i_1}), \ldots, 2d_G(v_{i_\ell}), 2d_G(v_{i_1}), \ldots, 2d_G(v_{i_\ell})\}. \tag{3}$$

If $t = 3$ or $t = 2$ so that $r = 0$, levels $t{-}4$ and $t{-}2$ above get replaced by levels 0 and 1 in $N(u_j^r)$. This yields the same degree multiset as in Equation 3.

Thus, equality of the multisets $D_{u_n^{t-1}}$ and $D_{u_j^r}$ gives equality of the sets in Equations 2 and 3. We can conclude that $\ell = k$. Furthermore, we have $2d_G(v_1) = 2d_G(v_{i_j})$ for some $i_j \in \{i_1, \ldots, i_\ell\}$. Proceeding inductively, we can reindex $\{1, \ldots, \ell\}$ if necessary so that $d_G(v_j) = d_G(v_{i_j})$. Thus, dropping these identical elements from each set, and using the equality gained from reindexing, we get:

$$\{d_G(v_1){+}1, \ldots, d_G(v_k){+}1\} = \{2d_G(v_1), \ldots, 2d_G(v_k)\}.$$

Using the same argument used in the proof of Lemma 3, we see that this is only possible if all k neighbors of v_n have degree 1 in G. However, if all neighbors of v_n in G have degree 1, then our assumption that G is connected requires that G be a star graph and that v be dominating in G. This contradicts Lemma $1'$.

We conclude then that if $t \geq 3$, then $\phi(u_n^{t-1}) \neq u_j^{t-3}$ and if $t = 2$, then $\phi(u_n^{t-1}) \neq u_j^0$, for any $1 \leq j \leq n$.

This finishes Case (I), so that $\phi(w) \neq u_n^{t-1}$.

Case (II): Suppose that $\phi(w) = u_n^s$ for some $0 \leq s \leq t{-}2$.

Since $d_G(v) = k$, and $s < t$, we have $d(u_n^s) = 2k$. Hence $\phi(w) = u_n^s$ gives $d(u_n^s) = d(w) = n$. We will show the equality $D_w = D_{\phi(w)} = D_{u_n^s}$ required by properties of automorphisms leads to a contradiction.

By construction, $N(w) = \{u_1^t, \ldots, u_n^t\}$ and $d(u_i^t) = d_G(v_i) + 1$. Thus

$$D_w = \{d_G(v_1){+}1, \ldots, d_G(v_n){+}1\}.$$

If $1 \leq s \leq t{-}2$, then $N(u_n^s) = \{u_1^{s-1}, \ldots, u_k^{s-1}, u_1^{s+1}, \ldots, u_k^{s+1}\}$, and since $d(u_i^{s+1}) = d(u_i^{s-1}) = 2d_G(v_i)$, we see that

$$D_{u_n^s} = \{2d_G(v_1), 2d_G(v_1), \ldots, 2d_G(v_k), 2d_G(v_k)\}.$$

If $s = 0$, level $s-1$ above gets replaced by level 0 in $N(u_n^s)$. This gives the same neighborhood degree multiset for $D_{u_n^s}$.

Thus equality of D_w and $D_{u_n^s}$ gives

$$\{d_G(v_1)+1, \ldots, d_G(v_n)+1\} = \{2d_G(v_1), 2d_G(v_1), \ldots, 2d_G(v_k), 2d_G(v_k)\}. \quad (4)$$

If there exists an i in $1 \leq i \leq k$ with $d_G(v_k) > 1$, let

$$d_{\min} = \min_{1 \leq i \leq k} \{d_G(v_i) : d_G(v_i) > 1\}.$$

Let j in be such that $d_G(v_j) = d_{\min}$. Then as $d_G(v_j)+1$ appears on the left hand side of Equation 4, there is a j' with $1 \leq j' \leq k$ such that $d_G(v_j)+1 = 2d_G(v_{j'})$. Because $d_G(v_j) = d_{\min} > 1$, we find $d_G(v_{j'}) > 1$. Then, by selection of $d_{\min}$ and that $1 \leq j' \leq k$, we have $d_G(v_j) \leq d_G(v_{j'})$. However, the equality $d_G(v_j) + 1 = 2d_G(v_{j'})$ and $d_G(v_j) > 1$ imply that $d_G(v_{j'}) < d_G(v_j)$. This contradiction lets us conclude $d_G(v_i) = 1$ for all $1 \leq i \leq k$.

But, then all neighbors of v_n in G have degree 1. Thus, our assumption that G is connected requires that G be a star graph with v_n be dominating in G. This contradicts Lemma $1'$.

Thus $\phi(w) \neq u_n^s$ for any $0 \leq s \leq t-2$. □

Lemma $2'$ leaves two possibilities for automorphisms that do not fix the shadow master.

One is that $|G| < 3$. Of these, K_1 is fully addressed by Lemma $3'$. Additionally, $K_1 + K_1$, where $+$ indicates disjoint union, is a disconnected graph, which is addressed by Lemma 4. Finally, we have $K_2 = K_{1,1}$. Here, we have $\mu^{(t)}(K_2) = C_{2t+3}$, a vertex-transitive graph. As we will see in Lemma $3'$, K_2 is the only star graph with automorphisms not mapping w to a top-level shadow vertex.

The other possibility is that G has an automorphism where w is mapped to a top-level shadow vertex. Lemma $3'$ shows that this only occurs when G is a star graph.

Lemma 3′. *If there is an automorphism ϕ of $\mu^{(t)}(G)$ that takes the shadow master w to a shadow vertex at level t, then $G = K_{1,m}$ for some $m \geq 0$. Additionally, if $|G| \neq 2$, then $\phi(w)$ is the shadow vertex at level t of the unique vertex of maximum degree in G.*

Proof. Let $|G| = n$ and let ϕ be an automorphism of $\mu^{(t)}(G)$ such that $\phi(w)$ is a shadow vertex at level t. Label the vertices so that $\phi(w) = u_n^t$. Then u_n^t is a shadow of v_n.

Suppose $n = 1$. Then $G = K_{1,0}$ and $\mu^{(t)}(G)$ is a set of isolated vertices, $\{u_1^0, u_1^1, \ldots, u_1^{t-1}\}$, together with a K_2 consisting of shadow vertex u_1^t and shadow master w. Clearly $\phi(w)$ must be u_1^t, the only other nonisolated vertex of $\mu^{(t)}(G)$.

Now, suppose $n > 1$. Since $u_n^t = \phi(w)$, we have $D_{u_n^t} = D_w$. As in Lemma 3, this allows us to conclude $G = K_{1,n-1}$.

By construction and properties of automorphisms, $n = d(w) = d(u_n^t) = d_G(v_n){+}1$. Thus, $d_G(v_n) = n-1$, so that $N_G(v_n) = \{v_1, \ldots, v_{n-1}\}$. Hence, $N(u_n^t) = \{u_1^{t-1}, \ldots, u_{n-1}^{t-1}, w\}$ and

$$D_{u_n^t} = \{d(u_1^{t-1}), \ldots, d(u_{n-1}^{t-1}), d(w)\} = \{2d_G(v_1), \ldots, 2d_G(v_{n-1}), d(w)\}.$$

On the other hand, by construction $N(w) = \{u_1^t, \ldots, u_n^t\}$. Thus,

$$D_w = \{d(u_1^t), \ldots, d(u_n^t)\} = \{d_G(v_1){+}1, \ldots, d_G(v_n){+}1\}.$$

In D_w, we have $d(w) = n$ and in $D_{u_n^t}$ we have $d(v_n){+}1 = n$, so after equating the two and removing $d(w) = d_G(v_n) + 1$, we get:

$$\{d_G(v_1){+}1, \ldots, d_G(v_{n-1}){+}1\} = \{2d_G(v_1), \ldots, 2d_G(v_{n-1})\}.$$

This is the same equation as Equation 1. Hence, as in the proof of Lemma 3, we can conclude that $G = K_{1,n-1}$. Then v_n is the unique vertex of maximum degree in G and $\phi(w)$ is its shadow at level t. □

4 Distinguishing Mycielskian Graphs

In Sections 2 and 3 we studied the action of an automorphism on $\mu^{(t)}(G)$. For convenience in the proof of Theorem 1, we combine Lemmas 2, 2′, 3, 3′, and 4, with the earlier observation about K_2, into a single lemma.

Lemma 5. *Let G be a graph and let $t \in \mathbb{N}$. Let ϕ be an automorphism of $\mu^{(t)}(G)$.*

- *If $G = K_{1,1} = K_2$, then $\mu^{(t)}(G) = C_{2t+3}$, and $\phi(w)$ can be any vertex.*
- *If $G = K_{1,m}$ for $m \neq 1$ then $\phi(w) \in \{w, u^t\}$, where u^t is the top-level shadow vertex of the vertex of degree m in $K_{1,m}$.*
- *If $G \neq K_{1,m}$ for any m, then $\phi(w) = w$.*

We are now ready to state and prove our main result which says that with few exceptions, $\text{Dist}(\mu^{(t)}(G)) \leq \text{Dist}(G)$. This proves Conjecture 1 in [4].

Theorem 1. *Let G be a graph with $\ell \geq 0$ isolated vertices and let $t \in \mathbb{N}$.*

- *If $G = K_1$, then $\text{Dist}(\mu(G)) = 2$, while for $t > 1$, $\text{Dist}(\mu^{(t)}(G)) = t$, exceeding $\text{Dist}(G) = 1$ for all t.*
- *If $G = K_2$, then $\text{Dist}(\mu(G)) = 3$, while for $t > 1$, $\text{Dist}(\mu^{(t)}(G)) = 2$, exceeding $\text{Dist}(G) = 2$ only for $t = 1$.*
- *If $t\ell > \text{Dist}(G)$, then $\text{Dist}(\mu^{(t)}(G)) = t\ell$, exceeding $\text{Dist}(G)$.*
- *Otherwise, if $G \neq K_1, K_2$ and $t\ell \leq \text{Dist}(G)$, then $\text{Dist}(\mu^{(t)}(G)) \leq \text{Dist}(G)$.*

Note that the last case covers nearly all graphs. For example, it covers all connected graphs with at least three vertices.

Proof. If $G = K_1$ then $\mathrm{Dist}(G) = 1$ and $\mathrm{Dist}(\mu(G)) = \mathrm{Dist}(K_1 + K_2) = 2$. When $t > 1$, since G has $\ell = 1$ isolated vertices, we have $t = t\ell > \mathrm{Dist}(G)$, and so this case is handled below.

If $G = K_2$, then $\mathrm{Dist}(G) = 2$. As already observed, $\mu^{(t)}(G) = C_{2t+3}$. Since $\mathrm{Dist}(C_5) = 3$ and $\mathrm{Dist}(C_n) = 2$ when $n \geq 6$, the result holds.

Let $|G| = n$ and G have $0 \leq \ell \leq n$ isolated vertices.

If $\ell > 0$, label the graph so that the isolated vertices are $v_1, \ldots, v_\ell$. By the generalized Mycielskian construction, $\mu^{(t)}(G)$ has a collection of $t\ell$ mutual twins $T = \bigcup_{i=0}^{t-1}\{u_1^i, \ldots, u_\ell^i\}$ consisting of isolated vertices and a set of ℓ mutual twins $U = \{u_1^t, \ldots, u_\ell^t\}$ consisting of degree-1 neighbors of w. For each $0 \leq s \leq t$, let R_s be the remaining vertices at level s, so that $R_s = \{u_{\ell+1}^s, \ldots, u_n^s\}$. Note if $\ell = n$, then R_s is empty for each $0 \leq s \leq t$. Similarly, if $\ell = 0$, let T and U be empty.

Suppose $t\ell > \mathrm{Dist}(G)$. If $\ell = 0$, then $t\ell = 0 < \mathrm{Dist}(G)$, so we may assume $1 \leq \ell \leq n$. Since mutual twins must receive distinct colors in a distinguishing coloring, $\mathrm{Dist}(\mu^{(t)}(G)) \geq |T| = t\ell$. We will now describe a $t\ell$-distinguishing coloring.

First, give each vertex in T a distinct color. For the vertices in U, give u_i^t the color of $u_i^0 = v_i$ for $1 \leq i \leq \ell$. Next, if $\ell < n$, use at most $\mathrm{Dist}(G) < t\ell$ colors on $\{v_{\ell+1}, \ldots, v_n\} = R_0$ so that the induced coloring on G is distinguishing and also color each shadow vertex u_j^s the same color as v_j for $1 \leq s \leq t$ and $\ell < j \leq n$. Finally, give w any of the $t\ell$ colors, other than the color on v_1 and u_1^t.

Now, let ϕ be an automorphism of $\mu^{(t)}(G)$ that respects this coloring. If $G = K_1$, then w and u_1^t having different colors means ϕ fixes w. Otherwise, the presence of isolated vertices means that G is not $K_{1,m}$ for any $m \geq 1$ and so, by Lemma 5, ϕ fixes w. Every vertex of T is fixed since these are the only vertices of degree 0 and each has a distinct color. Similarly, the vertices in U are the only vertices adjacent to w with degree 1, and each vertex of U has a different color, so ϕ fixes each vertex in U.

If R_s is nonempty, then by construction, for $0 \leq s \leq t$, the distance between w and vertices in R_s is $t - s + 1$. Since automorphisms preserve distances, the sets $R_0, \ldots, R_t$ are preserved by ϕ. Since the coloring of G is distinguishing, ϕ fixes each vertex in G, and therefore in the set $\{v_{\ell+1}, \ldots, v_n\} = R_0$. Suppose now that R_{s-1} is fixed pointwise. Since for $\ell + 1 \leq i \leq n$, we have $\phi(u_i^s) \in R_s$, let $\phi(u_i^s) = u_j^s$ for $i \neq j$. Since automorphisms preserve adjacency and R_{s-1} is fixed pointwise, we have $N(u_i^s) \cap R_{s-1} = N(\phi(u_i^s)) \cap R_{s-1} = N(u_j^s) \cap R_{s-1}$. By construction, this can only occur if v_i and v_j are twins. However, since the coloring restricted to G is distinguishing, v_i and v_j have different colors. Thus, in our coloring u_i^s and u_j^s received different colors, a contradiction. This shows that R_s must be fixed pointwise as well.

Thus, ϕ fixes every vertex of $\mu^{(t)}(G)$ and so we have $t\ell$-distinguishing coloring of $\mu^{(t)}(G)$. This shows that when $t\ell > \mathrm{Dist}(G)$, we have $\mathrm{Dist}(\mu^{(t)}(G)) = t\ell$.

For the remainder of the proof, we assume $G \neq K_1, K_2$ and $t\ell \leq \text{Dist}(G)$. We consider two cases based on whether the automorphism fixes the shadow master.

Suppose first that $\mu^{(t)}(G)$ has an automorphism that does not fix w. Since $G \neq K_1, K_2$, by Lemma 5, $G = K_{1,m}$ for some $m \geq 2$. Hence, $\text{Dist}(G) = m$. Let v_{m+1} be the unique vertex of degree m in G. By the structure of $K_{1,m}$ and $\mu^{(t)}(K_{1,m})$, we see that

- $d(u_{m+1}^s) = 2m$ for all $0 \leq s \leq t-1$ and $d(u_{m+1}^t) = m+1$;
- vertices $v_1, \ldots, v_m$ are mutually twin in $\mu^{(t)}(G)$ since each has neighborhood $\{v_{m+1}, u_{m+1}^1\}$;
- for each $1 \leq s \leq t$, vertices $u_1^s, \ldots, u_m^s$ are mutually twin in $\mu^{(t)}(G)$ with shared neighborhood $\{u_{m+1}^{s-1}, u_{m+1}^{s+1}\}$ when $s \neq t$ and $\{u_{m+1}^{t-1}, w\}$ when $s = t$.

Note that since $v_1, \ldots, v_m$ are mutually twin, each needs a distinct color in a distinguishing coloring. Therefore, $\text{Dist}(\mu^{(t)}(G)) \geq m$. We claim that, in fact, $\text{Dist}(\mu^{(t)}(G)) = m$.

Consider the following m-coloring of $\mu^{(t)}(G)$: for $1 \leq i \leq m$ assign color i to u_i^s, for $0 \leq s \leq t$. Assign color 1 to w and color 2 to u_{m+1}^s, for $0 \leq s \leq t$. Suppose that ϕ is an automorphism of $\mu^{(t)}(G)$ that preserves these color classes. Let C_i be the set of vertices with color i.

We have $C_1 = \{u_1^0, u_1^1, \ldots, u_1^t, w\}$. Since each vertex in $C_1 \setminus \{w\}$ has degree 2, while w has degree $m+1 > 2$, we have w is fixed by ϕ. Furthermore, since the distance from w to u_1^s is $t-s+1$, these unique distances from a vertex fixed by ϕ guarantee that C_1 is fixed pointwise by ϕ.

We have $C_2 = \{u_2^0, \ldots, u_2^t, u_{m+1}^0, \ldots, u_{m+1}^t\}$. The vertices in $\{u_2^0, \ldots, u_2^t\}$ have degree 2, while the vertices in $\{u_{m+1}^0, \ldots, u_{m+1}^t\}$ have degree $2m$ or $m+1$, each of which is strictly greater than 2. Therefore, ϕ fixes each setwise. Furthermore, as before, within each of these subsets, the vertices have distinct distances from the fixed vertex w. Thus, C_2 is also fixed pointwise by ϕ.

For each $3 \leq i \leq m$, we have $C_i = \{v_i, u_i^1, \ldots, u_i^t\}$. Again, the vertices of C_i have distinct distances from the fixed vertex w, and so C_i is fixed pointwise by ϕ.

Thus, this is an m-distinguishing coloring of $\mu^{(t)}(K_{1,m})$ when $m \geq 2$ so that $\text{Dist}(\mu^{(t)}(K_{1,m})) = \text{Dist}(K_{1,m})$ for $m \geq 2$. In particular, when $G \neq K_1, K_2$, $t\ell \leq \text{Dist}(G)$, and G has an automorphism that does not fix w, we have $\text{Dist}(\mu^{(t)}(G)) \leq \text{Dist}(G)$.

Finally, suppose that every automorphism of $\mu^{(t)}(G)$ fixes w. Recall that we have assumed $G \neq K_1, K_2$ and that $t\ell \leq \text{Dist}(G)$. Let $\text{Dist}(G) = k$ and fix a k-distinguishing coloring of G. We extend this coloring to a k-distinguishing coloring of $\mu^{(t)}(G)$.

First, color all original vertices in $\mu^{(t)}(G)$ with the k-distinguishing coloring of G. To be distinguishing, any twin vertices in G must receive different colors. In particular, if $\ell \geq 2$, the isolated vertices of G have distinct colors. As before, extend the coloring to the rest of the isolated vertices in T, giving each a distinct color. Since $|T| = t\ell \leq \text{Dist}(G)$, we have enough colors for this step. For vertices that are not isolated, color each shadow vertex u_j^s the same color as $u_j^0 = v_j$, for $1 \leq s \leq t$, $1 \leq j \leq n$. Finally, give w any of the k colors. We claim this is an k-distinguishing coloring of $\mu^{(t)}(G)$.

Let ϕ be an automorphism of $\mu^{(t)}(G)$ that respects this coloring. Since all vertices of T received different colors, ϕ fixes all isolated vertices. For $0 \leq s \leq t$, the sets $R_s = \{u_{\ell+1}^s, \ldots, u_n^s\}$ are nonempty. As before, the distance between vertices in R_s and w is a function of s. Since w is fixed, these sets are preserved setwise by ϕ. Also as before, our coloring of R_0 comes from a distinguishing coloring of G, so R_0 is fixed pointwise. An induction argument can again be used to show that this guarantees each set R_s is fixed pointwise, so that we have a distinguishing coloring of $\mu^{(t)}(G)$.

Thus, $\text{Dist}(\mu^{(t)}(G)) \leq k = \text{Dist}(G)$ when w is fixed and $t\ell \leq \text{Dist}(G)$. □

The following corollary is immediate from Theorem 1 since if G has ℓ isolated vertices then $\text{Dist}(G) \geq t\ell$ when $t = 1$. The corollary proves and exceeds the conjecture by Alikhani and Soltani.

Corollary 1. *For all graphs G with $G \neq K_1, K_2$,* $\text{Dist}(\mu(G)) \leq \text{Dist}(G)$.

In summary, for traditional Mycielskian graphs, the only exceptions are K_1 and K_2. We note that K_2 is an unsurprising exception since $\mu(K_2) = C_5$ is, in a sense, an exception among cycles, since it is the only cycle with distinguishing number 3 that is realizable as a Mycielskian graph. Furthermore, we proved that for generalized Mycielskian graphs with $t > 1$, the only exception is when $\mu^{(t)}(G)$ has so many isolated vertices that their number exceeds $\text{Dist}(G)$.

We note here that we have not proved that $\text{Dist}(G) = \text{Dist}(\mu^{(t)}(G))$. In fact, generalized Mycielskians of complete graphs show us that $\text{Dist}(G)$ and $\text{Dist}(\mu^{(t)}(G))$ may be arbitrarily far apart. We have $\text{Dist}(K_n) = n$ always. On the other hand, for $n \geq 3$, Proposition 1 below shows that $\text{Dist}(\mu(K_n)) = \lceil \sqrt{n}\, \rceil$. Additionally, if $n \geq 3$ and $t \geq \log_2 n - 1$, then $\text{Dist}(\mu^{(t)}(K_n)) = 2$. Using white as color 1 and red as color 2, Figure 3 shows the 2-distinguishing colorings described in Proposition 1 for $\mu(K_3)$ and $\mu^{(2)}(K_3)$.

Proposition 1. *Let $n \geq 3$ and $t \in \mathbb{N}$. Let $k \in \mathbb{N}$ be the least value satisfying $k^{t+1} \geq n$. Then* $\text{Dist}(\mu^{(t)}(K_n)) = k$.

Proof. Let k be the least value satisfying $k^{t+1} \geq n$. Since $k^{t+1} > n-1$, the base-k representation of $n-1$ has at most $t+1$ digits with each digit between 0 and $k-1$. For each $1 \leq i \leq n$, let r_i be the representation of $i-1$ in base k, with leading 0s appended so that r_i has $t+1$ digits.

We give a k-coloring of $\mu^{(t)}(K_n)$ as follows: give w color 1 and for $1 \leq i \leq n$ and $0 \leq s \leq t$, give u_i^s color $c+1$ if the $(s+1)$-st digit in r_i is c. Since $0 \leq c \leq k-1$, this is a k-coloring of $\mu^{(t)}(G)$. We will prove that this k-coloring is distinguishing.

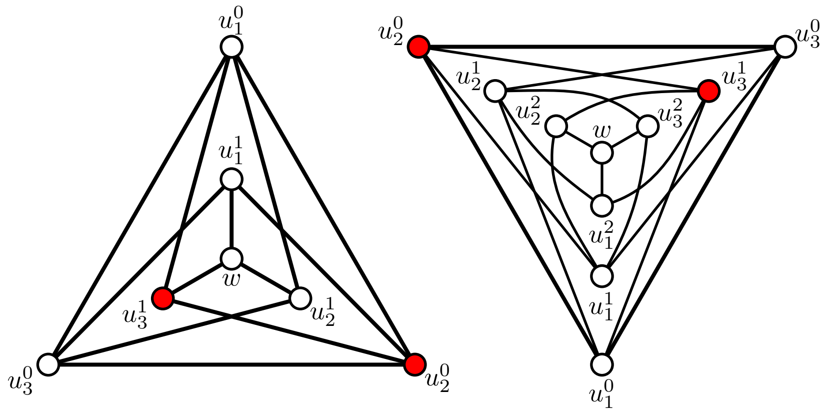

Figure 3: A 2-distinguishing coloring of $\mu(K_3)$ and $\mu^{(2)}(K_3)$.

Given any i and j, with $i \neq j$ there is an $\hat{s}$ with $0 \leq \hat{s} \leq t$ such that r_i and r_j are different in digit $\hat{s}+1$. Therefore, at level $\hat{s}$, vertices $u_i^{\hat{s}}$ and $u_j^{\hat{s}}$ receive different colors.

By Lemma 5, every automorphism of $\mu^{(t)}(K_n)$ fixes w. Since automorphisms preserve distances, the levels are fixed setwise by every automorphism. Moreover, by construction, for $1 \leq s \leq t$ and $1 \leq i \leq n$, the only non-neighbor of u_i^s at level $s-1$ is u_i^{s-1}. Since automorphisms preserve non-adjacency, $\phi(u_i^{\hat{s}}) = u_j^{\hat{s}}$ for some $\hat{s}$ if and only if $\phi(u_i^s) = u_j^s$ for all $0 \leq s \leq t$.

However, we have shown for each $i \neq j$ there exists an $\hat{s}$ where the colors on $u_i^{\hat{s}}$ and $u_j^{\hat{s}}$ differ. Thus, to preserve the color classes, an automorphism ϕ must have $\phi(u_i^s) = u_i^s$ for all $1 \leq i \leq n$ and $0 \leq s \leq t$. Thus, this coloring is k-distinguishing and so $\text{Dist}(\mu^{(t)}(K_n)) \leq k$.

Let $\ell \in \mathbb{N}$ such that $\ell < k$. Since k is the least value satisfying $k^{t+1} \geq n$, it must be the case that $\ell^{t+1} < n$. We claim there does not exist an ℓ-distinguishing coloring of $\mu^{(t)}(K_n)$.

There are at most ℓ^{t+1} lists of the form $(c_0, \ldots, c_t)$ with $1 \leq c_s \leq \ell$ for each $0 \leq s \leq t$. Hence, by Pigeonhole Principle, in any ℓ-coloring of $\mu^{(t)}(K_n)$, there exist distinct i and j such that the colors of u_i^s and u_j^s agree for each $0 \leq s \leq t$. Then, the automorphism ϕ with $\phi(u_i^s) = u_j^s$ and $\phi(u_j^s) = u_i^s$ for each $0 \leq s \leq t$ and $\phi(x) = x$ for all other vertices x, preserves the color classes. Hence, there does not exist an ℓ-distinguishing coloring of $\mu^{(t)}(K_n)$ for all $\ell < k$. It follows that $\mu^{(t)}(K_n) \geq k$ and, therefore, $\mu^{(t)}(K_n) = k$.

□

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