Structure And Properties Of Locally Outerplanar Graphs

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Abstract

This paper studies convex geometric graphs in which no path of length 3 self-intersects. A main result gives a decomposition of such graphs into induced outerplanar graph drawings. The resulting structure theorem is then used to compute a sharp, linear upper bound on the size of the edge set in terms of the number of vertices and the number and type of graphs in the decomposition. The paper also shows that though locally outerplanar graphs have hereditary properties, no graph property that is closed under the taking of minors can hold for all locally outerplanar graphs. Each of these results is generalized to convex geometric graphs in which no path of length k self-intersects.

1 Introduction

A straight-line graph drawn in the plane with vertices in general position is called a geometric graph. Much of the work in geometric graph theory is extremal in nature. The canonical question is: "What is the maximum number of edges that a geometric graph on n vertices can have without containing a given geometric subgraph?" Some of the forbidden subgraphs that have been studied are: sets of pairwise disjoint edges, sets of pairwise crossing edges, noncrossing cycles, self-intersecting cycles, noncrossing paths, and self-intersecting paths. A survey of these and other results in geometric graph theory is provided in [4].

The forbidden subgraphs we focus on here are self-intersecting paths. A geometric graph with no self-intersecting path of length 3 is called *locally planar* and such a graph with vertices in convex position is called *locally outerplanar*. Pach, Pinchasi, Tardos and Tóth [5] proved that locally planar graphs have at most $O(n \log n)$ edges and that this bound is asymptotically tight. In contrast Boutin in [1] and Brass, Károlyi and Valtr in [2] independently proved that a locally outerplanar graph has at most $2n-3$ edges. Additionally, Boutin proved that if such a graph has at least one crossing, it has no more than $2n-6$ edges. We will see a generalization of this bound in Corollary 1.2. In [6], Tardos constructs geometric graphs with no self-intersecting path of length $2k+1$ that have $\theta(n \log^{(k)} n)$ edges, where $\log^{(k)}$ is the k-times-iterated log function. Corollary 2.1 will provide contrast with this result.

A main theorem in this paper addresses the structure of locally outerplanar graphs. We can easily create these graphs using vertex disjoint outerplanar graph drawings (called layering subgraphs) and connecting them pairwise with additional outerplanar graph drawings (called tethering subgraphs) by vertices at distance at least 2. With just a little care in construction, the result is a locally outerplanar graph. Figure 1 illustrates. The layering subgraphs have edges drawn in solid lines; the tethering subgraphs have edges drawn in dotted lines.

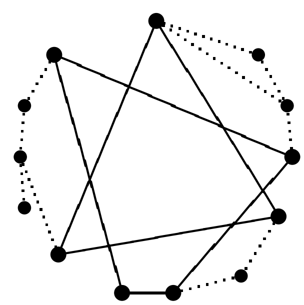

Figure 1:

Previous work suggests that a graph is locally outerplanar graphs if and only if it can be constructed in this manner. This is true but the formal statement, Theorem 1, requires more detail. This structure theorem is then used to generalize the edge bound found in [1]; it provides a sharp linear upper bound on the number of edges in a locally outerplanar graph in terms of its number of vertices, layering subgraphs, and tethering subgraphs.

The property of being a locally outerplanar graph is a hereditary property (that is, a property that is closed under the taking of induced subgraphs). In previous work [1], we've seen that locally outerplanar graphs are 3-colorable and have vertex and edge arboricity 2. These are also hereditary properties. In contrast, Theorem 3 shows that every abstract graph is the minor of some locally outerplanar graph. Thus we can conclude that no property that is closed under the taking of minors (such as bounded tree width) can hold for all locally outerplanar graphs.

The structure, edge bound, and hereditary results are generalized to convex geometric graphs in which no path of length k self-intersects (k-locally outerplanar). The bound on the number of edges in a k-locally outerplanar graph contrasts with results of Tardos on $(2k+1)$-locally planar graphs as mentioned earlier.

This paper is organized as follows: Section 2 provides notation and terminology that will be used in the remainder of the paper. Section 3 gives the statement and proof of the structure theorem for locally outerplanar graphs and the resulting edge bound. Section 4 generalizes these results to k-locally outerplanar graphs. Section 5 discusses hereditary properties and shows how to construct a locally outerplanar graph with a given minor.

2 Background on Local Outerplanarity

A *geometric graph* is a straight-line graph drawn in the plane in which no three vertices lie on a single line and no three edges intersect at a single point. A *convex geometric graph* is a geometric graph all of whose vertices lie on the boundary of its convex hull. We will use the notation $\overline{G}$ to denote a geometric graph and reserve the notation G for the underlying abstract graph. A geometric graph with no self-intersecting path of length 3 is called *locally planar*. A convex geometric graph with the same property is called *locally outerplanar*. The smallest graphs that are locally outerplanar but not outerplanar are subdivisions of $K_{2,3}$ and of K_4. See Figure 2.

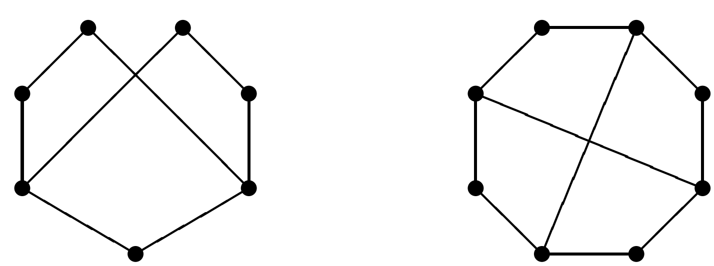

Figure 2:

The following notation and observations will be useful in the proof of the main theorem of of the paper. Some of these are covered in more detail in [1]. Let $\overline{G}$ be a convex geometric graph on n vertices. Label the vertices $1, \cdots, n$ moving clockwise around the convex hull. Using cyclic interval notation, let $[a, b]$ denote the vertices of $\overline{G}$ that lie clockwise between vertices a and b inclusive. Let (a, b) denote the vertices that lie clockwise strictly between a and b. Define the *cyclic distance* from a to b to be the smaller of the number of vertices in $(a, b]$ or in $(b, a]$, (This would be the length of a shortest path along the convex hull from a to b if all edges existed on the convex hull.) A pair of vertices is said to be *consecutive* on the convex hull if their cyclic distance is 1.

A path P of length m is given by an ordered set of $m + 1$ distinct vertices $a_0 a_1 \cdots a_m$ where each $a_j a_{j+1}$ is an edge of $\overline{G}$. We say a path is a *simple self-intersecting path* if its first and last edges cross and these are the only edges of the path that cross. That is, $P = a_0 \cdots a_m$ is a simple self-intersecting path if $a_0 a_1$ crosses $a_{m-1} a_m$ and $P' = a_1 \cdots a_{m-1}$ does not cross itself. Since P' does not cross itself or the edges $a_1 a_2$, $a_{m-1} a_m$, the vertices of P' are cyclically ordered around the outside of the convex hull. (That is, if a_0 is larger than all other a_i then either $a_1 < \cdots < a_{m-1}$ or $a_{m-1} < \cdots < a_1$.) We will assume that P is written so that P' is traversed clockwise.

Call an ordered pair of vertices $\langle u, v \rangle$ a *corner pair* if there is a simple self-intersecting path $P = aP'z$ so that u is the initial point of P' and v is the terminal point of P'. In particular at least one edge incident to u (*e.g.* au) crosses at least one edge incident to v (*e.g.* vz). Call the set of edges of $\overline{G}$ that have initial point u or v and cross edges with initial point v or u the *crossing edges at u and v*.

Define $\overline{H}_{u,v}$ to be the connected component containing u (and therefore v) of the subgraph induced by the vertices in $[u, v]$. (Recall that we are assuming that P' is traversed clockwise and that we know its vertices are cyclically ordered around the outside of its convex hull; therefore its vertices are contained in $[u, v]$.) Call a corner pair $\langle u, v \rangle$ a *minimal corner pair* if it is the only corner pair within $\overline{H}_{u,v}$. Then we can call u and v the *corner vertices* of $\overline{H}_{u,v}$ and all other vertices of $\overline{H}_{u,v}$ the *noncorner vertices* of $\overline{H}_{u,v}$. Call a simple self-intersecting path a *minimal self-intersecting path* if its corner pair is minimal. Notice that there may be many minimal self-intersecting paths with the same minimal corner pair.

Example 1. Consider the partial convex geometric graph in Figure 3. The path $P = hbcfga$ is one of the minimal self-intersecting paths with minimal corner pair $\langle b, g \rangle$. $\overline{H}_{b,g}$ is the subgraph induced by $\{b, c, d, f, g\}$

and is drawn in solid lines. The corners vertices of $\overline{H}_{b,g}$ are b, g while the noncorner vertices are c, d, f. The edges ag and bh are the crossing edges at g and b. Notice that the vertex e is not in $\overline{H}_{b,g}$ because it is not "local" to the paths we are considering here. Further notice that $\overline{H}_{b,g}$ is an outerplanar graph drawing - this fact will be proved later.

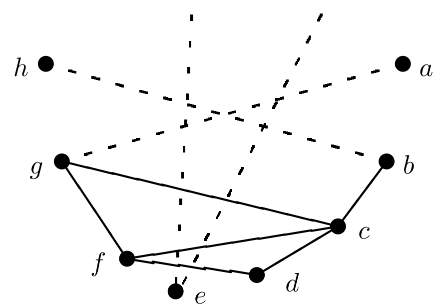

Figure 3:

3 Structure

A simple statement of the result we are looking for might be "A convex geometric graph is locally outerplanar if and only if it can be built by layering vertex disjoint outerplanar graph drawings and tying these together with nonadjacent vertices from additional outerplanar graph drawings." That is certainly the spirit of the result, but more detail is necessary to ensure that nothing "silly" is done when tying together the layering subgraphs and that nothing extraneous is implied. Once new terms are established Corollary 1.1 gives an easier way to state the result.

Theorem 1. A geometric graph $\overline{G}$ is locally outerplanar if and only if it contains two sequences of induced convex geometric subgraphs $\overline{H}_1, \cdots, \overline{H}_s$ and $\overline{G} = \overline{G}_0, \overline{G}_1, \cdots, \overline{G}_s$ so that $\overline{G}_s$ is a union of vertex disjoint outerplanar graph drawings, and for each $i \geq 1$

1. $\overline{H}_i$ and $\overline{G}_i$ provide an edge partition of $\overline{G}_{i-1}$
2. $\overline{H}_i$ is an outerplanar graph drawing
3. $V(\overline{G}_i) \cap V(\overline{H}_i) = \{u_i, v_i\}$ where
   (a) $V(\overline{H}_i) \subseteq [u_i, v_i]$ as a subset of $V(\overline{G})$

(b) u_i and v_i are consecutive vertices on the convex hull of $\overline{H}_i$ and the shortest path between them has length at least 2

(c) in $\overline{G}_i$, $[u_i, v_i]$ does not contain the terminal vertex of any edge with initial vertex in the first neighborhood of u_i or v_i.

Given the situation described above, we will call the subgraphs $\overline{H}_1, \cdots, \overline{H}_s$ the *tethering subgraphs* of $\overline{G}$ and the components of $\overline{G}_s$ the *layering subgraphs* of $\overline{G}$. We will call the vertices $V(\overline{H}_i) \cap V(\overline{G}_i)$ the *attaching vertices* of $\overline{H}_i$.

Proof. $\Longrightarrow$

Let $\overline{G}$ be a locally outerplanar graph. Suppose that the theorem is true for every locally outerplanar graph on fewer vertices than $\overline{G}$.

If $\overline{G}$ is an outerplanar graph drawing we are done.

If $\overline{G}$ is not an outerplanar drawing then it has crossing edges which implies the existence of a simple self-intersecting path, a minimal self-intersecting path, and therefore a minimal corner pair $\langle u, v\rangle$. Let $\overline{H}_1 = \overline{H}_{u,v}$, (the subgraph containing u (and therefore v) of the connected component of the graph induced by the vertices of $\overline{G}$ in $[u, v]$). Let $\overline{G}_1$ be the subgraph that results when we remove the edges and noncorner vertices of $\overline{H}_1$ from $\overline{G}$.

Since $\langle u, v\rangle$ is a minimal corner pair, there is no edge from a noncorner vertex of $\overline{H}_1$ to a vertex of (v, u); such an edge would necessarily cross one of the crossing edges at u and v, providing an additional corner pair in $\overline{H}_{u,v}$ and contradicting the minimality of $\langle u, v\rangle$. Also there is no edge from a noncorner vertex of $\overline{H}_1$ to a vertex outside of $\overline{H}_1$ within (u, v); such a vertex would necessarily belong to $\overline{H}_1$. Thus there are no edges between vertices outside of $\overline{H}_1$ and vertices outside of $\overline{G}_1$. This means that both $\overline{H}_i$ and $\overline{G}_i$ are induced graphs and that they provide an edge partition for $\overline{G}$. Thus Property 1 is satisfied.

If $\overline{H}_1$ contains crossing edges then there must be another pair of corner vertices in $\overline{H}_1$ - contradicting the minimality of $\langle u, v\rangle$. Thus $\overline{H}_1$ is an outerplanar graph drawing and Property 2 is satisfied.

The vertices of $\overline{H}_1$ are a subset of $[u, v]$ by definition so Property 3(a) is satisfied. Note that as a subset of $[u, v]$ the vertices of $\overline{H}_1$ are in convex position. Thus u and v are consecutive on the convex hull of $\overline{H}_1$. Since there are crossing edges incident to u and v and $\overline{G}$ is locally outerplanar, u and v are not adjacent. Thus Property 3(b) is satisfied. Note that by definition of $\overline{H}_1$ any edge from u to $[u, v]$ is in $\overline{H}_1$ and therefore not in $\overline{G}_1$. Further, if an edge from the first neighborhood of u or v (in $\overline{G}_1$) has terminal point in $[u, v]$ it must cross the crossing edges at u and v creating a crossing path of length 3. This cannot happen and Property 3(c) is satisfied.

By the inductive hypothesis, since $\overline{G}_1$ is an outerplanar graph on fewer vertices than $\overline{G}$, it can be decomposed yielding sequences $\overline{H}'_1, \cdots, \overline{H}'_{s-1}$ and $\overline{G}'_1, \ldots, \overline{G}'_{s-1}$ with appropriate properties. Adding $\overline{H}_1$ and $\overline{G}_1$ to the beginning of this sequence renumbering the rest as appropriate gives us the necessary sequences of induced outerplanar subgraph drawings.

$\Longleftarrow$

As an inductive hypothesis suppose that whenever we have two induced subgraph sequences of length $s-1$ satisfying the hypotheses of Theorem 1, $\overline{G}_0$ is a locally outerplanar graph.

Suppose that $\overline{G}$ is a convex geometric graph with induced subgraph sequences $\overline{H}_1, \cdots, \overline{H}_s$ and $\overline{G} = \overline{G}_0, \overline{G}_1, \cdots, \overline{G}_s$ satisfying the hypotheses of Theorem 1.

Drop $\overline{H}_1$ and $\overline{G}_0$ from the given sequences. This leaves us with two sequences of length $s-1$ meeting all appropriate requirements. Thus the inductive hypothesis gives us that $\overline{G}_1$ is locally outerplanar. We will see that this assumption and the hypotheses on $\overline{H}_1$ and $\overline{G}_1$ give us that $\overline{G}_0 = \overline{G}$ is a locally outerplanar graph.

Suppose not. Then there is a self-intersecting path P of length 3 in $\overline{G}_0$ that is not in $\overline{G}_1$. Then at least one of the edges of P is in $\overline{H}_1$.

Notice that $\overline{H}_1$ cannot provide the middle edge of P because this would mean that the incident vertices of this edge are both adjacent and the attaching vertices of $\overline{H}_1$ and this would violate Property 3b). Then the uncrossed edge of P is an edge of $\overline{G}_1$. Since both $\overline{G}_1$ and $\overline{H}_1$ are outerplanar graph drawings exactly one of the crossing edges of P lives in each of $\overline{G}_1$ and $\overline{H}_1$. Suppose that $P = abcd$ so that bc and cd are edges of $\overline{G}_1$ and that ab is an edge of $\overline{H}_1$. In particular, b is a vertex of both $\overline{G}_1$ and $\overline{H}_1$. Thus b is an attaching vertex of $\overline{H}_1$ and without loss of generality $b = u_1$. Then since $bcd = u_1cd$ is a path, cd is an edge of $\overline{G}_1$ with one vertex in the first neighborhood of u_1. Since cd crosses an edge of $\overline{H}_1$ we know that either c or d falls inside (u_1, v_1). Since c is adjacent to u_1 if it were inside (u_1, v_1) it would be in $\overline{H}_1$. Thus d is in (u_1, v_1) and $\overline{G}_1$ has an edge (cd) with one vertex in the first neighborhood of u_1 and another vertex in (u_1, v_1). This contradicts property 3(c), and therefore cannot happen. Thus there is no crossing path of length 3 in $\overline{G} = \overline{G}_1 \cup \overline{H}_1$ and therefore $\overline{G}$ is locally outerplanar. □

More generally, suppose that $\overline{H}$ is an induced outerplanar subgraph drawing of an arbitrary convex geometric graph $\overline{G}$. Let $\overline{G}'$ be the convex geometric graph we get from $\overline{G} - E(\overline{H})$ by discarding any isolated vertices. If $\overline{G}'$ and $\overline{H}$ fulfill properties 3(a), 3(b) and 3(c) of Theorem 1 then we call $\overline{H}$ a *tethering subgraph* of $\overline{G}$ (even though $\overline{G}$ may not be locally outerplanar). This allows us to state our structure result more succinctly.

Corollary 1.1. A convex geometric graph is locally outerplanar if and only if we can recursively remove the edges and nonattaching vertices of tethering subgraphs until we are left with vertex disjoint outerplanar graph drawings.

The following corollary generalizes Theorem 2 of [1] which states that a locally outerplanar graph on n vertices with at least one crossing has at most $2n - 6$ edges.

Corollary 1.2. A locally outerplanar graph with n vertices and r layering subgraphs has at most $2n - 3r$ edges.

Proof. Each layering subgraph provides at most three fewer edges than the double of its vertices. Thus if the locally outerplanar subgraph consisted strictly of r layering subgraphs it would have at most $2n - 3r$ edges. A tethering subgraph on m vertices adds $m - 2$ new vertices to $\overline{G}$ and since it is nonmaximal, it contributes at most $2m - 4$ new edges. Thus it contributes at most twice as many new edges as new vertices, thus maintaining the proportion. □

4 Generalization

The above results can be quickly generalized to convex geometric graphs with no crossing path of length k. Such a graph will be called *k-locally outerplanar*.

The only difference in the decomposition of a k-locally outerplanar graph is that the nonmaximal outerplanar graph drawings $\overline{H}_{u,v}$ are more than one edge away from being maximal. The attaching vertices u and v are consecutive on the convex hull of $\overline{H}_{u,v}$ but the shortest path between these minimal corner vertices has length $k - 1$.

Theorem 2. A geometric graph $\overline{G}$ is k-locally outerplanar if and only if it contains two sequences of induced subgraphs $\overline{H}_1, \cdots, \overline{H}_s$ and $\overline{G} = \overline{G}_0, \overline{G}_1, \cdots, \overline{G}_s$ so that $\overline{G}_s$ is a union of vertex disjoint outerplanar graph drawings, and for each $i \geq 1$

1. $\overline{H}_i$ and $\overline{G}_i$ provide an edge partition of $\overline{G}_{i-1}$

2. $\overline{H}_i$ is an outerplanar graph drawing

3. $V(\overline{G}_i) \cap V(\overline{H}_i) = \{u_i, v_i\}$ where

    (a) $V(\overline{H}_i) \subseteq [u_i, v_i]$ as a subset of $V(\overline{G})$

    (b) u_i and v_i are consecutive vertices on the convex hull of $\overline{H}_i$ and the shortest path between them has length at least $k-1$

    (c) in $\overline{G}_i$, $[u_i, v_i]$ does not contain the terminal vertex of any edge with initial vertex in the first neighborhood of u_i or v_i.

This structure theorem also yields a bound on the maximum number of edges.

Corollary 2.1. A k-locally outerplanar graph with n vertices, r layering subgraphs, and s tethering subgraphs has at most $2n - 3r - (k-3)s$ edges.

Proof. Again each layering subgraph provides at most three fewer edges than the double of its vertices. Further, since the shortest path from u to v in $\overline{H}_{u,v}$ contains k vertices (including u and v), if we consider the abstract maximal outerplanar graph on these vertices, it would contain $2k-3$ edges, and only $k-1$ of them exist in $\overline{G}$ and therefore $\overline{H}_{u,v}$. Thus there are at least $2k-3-(k-1) = k-2$ fewer edges in $\overline{H}_{u,v}$ than in a maximal outerplanar graph on the same vertex set. Thus a tethering subgraph on m vertices adds $m-2$ vertices to $\overline{G}$ and at most $2m-3-(k-2) = 2(m-2)-(k-3)$ edges. Thus each tethering subgraph provides $(k-3)$ fewer edges than twice its number of vertices. Thus $\overline{G}$ has at most $2n-3r-(k-3)s$ edges. □

5 Hereditary Properties

In previous work [1] we saw that locally outerplanar graphs are 3-colorable and have both edge and vertex arboricity 2. However, Corollary 3.2 will show that locally outerplanar graphs do not have bounded tree width. More specifically, though the class of locally outerplanar graphs can have hereditary properties (properties that are closed under the taking of subgraphs or induced subgraphs), this class cannot have properties that are closed under the taking of minors. Theorem 3 shows this by proving that every graph is a minor of some locally outerplanar graph.

It is easy to see that any graph H is a minor of some locally planar graph. Take a geometric graph $\overline{H}$ whose underlying abstract graph is H. Subdivide each edge with a vertex of degree 2 yielding $\overline{H}'$. Any crossing path of length 3 in $\overline{H}$ becomes a crossing path of length 4 in $\overline{H}'$. Then $\overline{H}'$ is locally planar and has H as a minor.

It is not as immediate to see that every graph is a minor for some locally outerplanar graph. We not only need to subdivide certain edges of a convex geometric graph, but we must move the new vertices to the convex hull in a way that doesn't create (too many) new crossing 3 paths. This, however, can always be done and is proved in the following.

Theorem 3. *Every abstract graph is a minor for some locally outerplanar graph.*

Proof. Let H be an arbitrary abstract graphs and let $\overline{H}$ be a convex geometric graph whose underlying abstract graph is H.

If $\overline{H}$ is locally outerplanar we are done. If not, then find a crossing path of length 3, say $P = abcd$ in which the cyclic distance between the interior vertices b and c, is minimal. That is, the uncrossed edge bc of P is as close to the convex hull as any other uncrossed edge of a crossing 3 path.

The strategy here is to subdivide the edge bc and move the new vertex to the convex hull. This will eliminate (at least) one crossing path of length 3, but may create (at most) one new one. But any new crossing path of length 3 will have its uncrossed edge closer to the convex hull than the original. Since this can only be done a finite number of times before the number of crossing 3 paths is reduced, we repeat and eventually reduce the number of crossing paths of length 3. Then we can use an induction argument to complete the proof. The details follow.

Assume that the vertices of P lie on the convex hull clockwise in the order a, d, b, c. Subdivide bc by a vertex of degree 2 labeled x then redraw with x consecutive to b on the convex hull in a clockwise direction. See Figure 4 for an illustration. Call this new geometric graph $\overline{H}'$. This changes $abcd$, a crossing path of length 3 in $\overline{H}$, to $abxcd$, a crossing path of length 4 in $\overline{H}'$. We have removed the edge bc, but we've created two new edges bx and xc, and these may have created new edge crossings and therefore may have created new crossing 3 paths.

Since x is consecutive to b on the convex hull of $\overline{H}'$, no edge crosses bx. Thus any newly created crossings must involve the edge xc. A "new" edge crossing involving xc would be an edge that crosses xc in $\overline{H}'$ but did not cross bc in $\overline{H}$. Such an edge must be of the form by where $y \in (x, c)$ and would yield a crossing 3 path $cxby$. This crossing 3 path can be eliminated by subdividing the edge xb. If necessary do so and assume this vertex as part of $\overline{H}'$. We also get a new crossing 3 path $cxyb$ if there is an edge yx; but this cannot occur since x has degree 2. We also get a new crossing 3 path $cybx$ if there is an edge cy. These are the only ways that a new

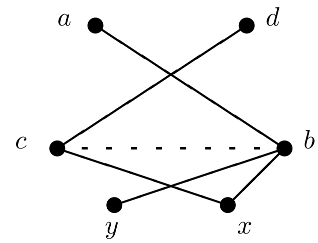

Figure 4:

crossing can yield a new crossing 3 path. Notice that $cybx$ has a smaller distance between the vertices of its uncrossed edge than did $abcd$ in $\overline{H}$.

Thus if we have one vertex y in (b, c) that is adjacent to both b and c we have a new crossing 3 path and the cyclic distance between the vertices of its uncrossed edge is strictly smaller than the original minimum. There cannot be two vertices y and z in (b, c) that both are adjacent to both b and c because would mean a crossing 3 path say $zycb$ with a smaller distance between the vertices of its uncrossed edge than did $abcd$ in $\overline{H}$.

Thus we have eliminated one or more crossing 3 paths of $\overline{H}$ and introduced at most one crossing 3 path in $\overline{H}'$ but one whose cyclic distance between vertices of its uncrossed edge is smaller than the minimum in $\overline{H}$. Thus if we repeat this process, after a finite number of steps we must have created a new convex geometric graph with strictly fewer crossing 3 paths than the original. At this stage we can appeal to induction to subdivide the remainder of the crossing 3 paths.

Thus we have found a subdivision and redrawing of $\overline{H}$ that is locally outerplanar. This locally outerplanar graph has H as a minor.

□

Corollary 3.1. *No graph property that is closed under taking minors is held by all locally outerplanar graphs.*

Since bounded tree width is a property that is closed under taking minors [3], we get the following.

Corollary 3.2. *Locally outerplanar graphs do not have bounded tree width.*

The proof of Theorem 3 can easily be generalized to show that every abstract graph is a minor for some k-locally outerplanar graph. The corollaries also generalize.

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