The Isometry Dimension and Orbit Number of a Finite Group

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Abstract

A finite set $W \subset \mathbb{R}^d$ is said to *realize* the group G if the isometry group of W is isomorphic to G. The *isometry dimension* of a group is the minimum dimension of a realization. It is known that the isometry dimension of G is less than $|G|$ [1]. We show that the isometry dimension of $\mathbb{Z}_2^n$ is n. The *orbit number* of a group is the minimum number of orbits in a realization. We show that the groups $\mathbb{Z}_2^n$ are the only abelian groups with orbit number 1. We provide examples that illuminate these parameters.

1 Introduction

A finite set $W \subset \mathbb{R}^d$ is said to *realize* the group G if $Iso(W) \cong G$. Here $Iso(W)$ is the isometry group of W, the bijections from W to itself that preserve pairwise distance. We say that the realization W has dimension d, denoted by $dim(W) = d$. It is natural to investigate the minimum d such that G has a realization of dimension d. This is called the *isometry dimension* of G, denoted by $\delta(G)$. Using the Implicit Function Theorem, Albertson and Boutin [1] showed that $\delta(G) < |G|$. Making an algebraic connection, Patnaik [2] showed that $\delta(G)$ equals the minimum degree of a faithful real representation of G. This strengthens the previous result since the regular representation of degree $|G|$ given by Cayley's Theorem is faithful, and one of its irreducible constituents is the identity. Subsequently Reichard [3] and Tucker [4] independently discovered this result.

Given a group G and a realization W, $dim(W)$ is just one of the parameters of interest. The *size* of the realization is $|W|$. Given a group G it is natural to look for realizations of minimum size. Consequently we define the *isometry size* $\nu(G) = \min\{|W| : W$ realizes $G\}$. Similarly the number of orbits in a realization is denoted by $orb(W)$, and we define the *orbit number* of G to be $\rho(G) = \min\{orb(W) : W \text{ realizes } G\}$.

Examples:

1. The endpoints of the standard basis vectors in $\mathbb{R}^n$ give a realization of S_n. The size and dimension are n and there is one orbit. Translating so that one of the points is at the origin drops the dimension to $n-1$. This translated realization of S_n simultaneously achieves the isometry dimension [2], the isometry size, and the orbit number.

2. The vertices of a regular n-gon in $\mathbb{R}^2$ give a realization of D_n of size n, dimension 2, with one orbit. This realization minimizes the dimension and orbit number for all $n > 2$, but does not always minimize the size. Note that $D_6 \cong D_3 \times \mathbb{Z}_2$ and in this latter form there is a realization with dimension 3, two orbits, and size 5.

3. Any two points in $\mathbb{R}$ realize $\mathbb{Z}_2$. For $n > 2$ it is natural to realize $\mathbb{Z}_n$ with $2n$ points in $\mathbb{R}^2$. Begin with n points that are the vertices of a regular n-gon. Between each pair of consecutive points of the polygon insert an orientation point so that the rotation symmetries remain but the reflection symmetries do not [1]. Thus it is clear that for $n > 2$, the isometry dimension of $\mathbb{Z}_n$ is 2. We show in Section 3 that if $n > 2$, the orbit number of $\mathbb{Z}_n$ is 2. However the isometry size of $\mathbb{Z}_n$ might be considerably less than $2n$. For example since $\mathbb{Z}_{35} \cong \mathbb{Z}_5 \times \mathbb{Z}_7$, we can realize $\mathbb{Z}_{35}$ in $\mathbb{R}^4$ with four orbits using 10 points that are zero in the second two coordinates and 14 points that are zero in the first two coordinates.

4. For our final example we will look at two realizations of $\mathbb{Z}_2^n$. The first is a generalized octahedron. Let

$$W = \{w_{i,j} : 1 \leq i \leq n, j \in \{1, -1\}\}.$$

Here $w_{i,j}$ denotes the point in $\mathbb{R}^n$ that has $i \cdot j$ in the i^{th} coordinate and 0 in all other coordinates. So when $n = 3$,

$$W = \{(1,0,0), (-1,0,0), (0,2,0), (0,-2,0), (0,0,3), (0,0,-3)\}.$$

Clearly W has n orbits, dimension n, and size $2n$. This realization achieves both the isometry size and the isometry dimension of $\mathbb{Z}_2^n$

(see Section 2). However there is a *dual* realization as a generalized cube which has just one orbit. Let

$$\hat{W} = \{\sum_{i=1}^{n} w_{i,\sigma(i)} : \sigma : \mathbb{Z}_n \to \{1, -1\}\}.$$

So when $n = 3$,

$$\hat{W} = \{(1,2,3), (1,2,-3), (1,-2,3), \ldots, (-1,-2,-3)\}.$$

Looking at the definition we can see that there are as many points in $\hat{W}$ as there are possible functions σ. Therefore, $|\hat{W}| = 2^n$. Clearly the dimension of $\hat{W}$ is n. Since $\mathbb{Z}_2^n$ acts on $\hat{W}$ in the natural way *viz.* the i^{th} generator changes the sign of the i^{th} coordinate, $\hat{W}$ has just 1 orbit.

The purpose of this note is

- to introduce the isometry size and orbit number of a group;
- to determine the isometry dimension of $\mathbb{Z}_2^n$;
- to construct realizations of groups with just one orbit; and
- to characterize the abelian groups with orbit number 1.

2 The Isometry Dimension of $\mathbb{Z}_2^n$

Theorem 1. *The isometry dimension of $\mathbb{Z}_2^n$ is n.*

Proof. Let $\mathbb{Z}_2^n = \langle \alpha_1, \ldots, \alpha_n : \alpha_i^2 = 1, \alpha_i\alpha_j = \alpha_j\alpha_i \rangle$. For the purpose of this proof, by $\mathbb{Z}_2^{n-1}$ we shall mean the subgroup of $\mathbb{Z}_2^n$ generated by $\alpha_1, \ldots, \alpha_{n-1}$.

The realizations of $\mathbb{Z}_2^n$ in the introduction both have dimension n. Thus $\delta(\mathbb{Z}_2^n) \leq n$. We use induction to show that if $\mathbb{Z}_2^n$ acts by isometries on a set of points in $\mathbb{R}^m$ then the span of these points contains an n-cube and therefore must have dimension at least n.

Base Case: If $\mathbb{Z}_2$ acts on a set of points in $\mathbb{R}^n$ there is some point z moved by the generator α. Then z and $\alpha(z)$ form a 1-cube.

The induction hypothesis: If X is a set of points whose isometry group contains a subgroup isomorphic to $\mathbb{Z}_2^{n-1}$ then within the span of X is a point y that has a distinct image under each element of $\mathbb{Z}_2^{n-1}$. Further, whenever $i \neq j$ the line segments $[y, \alpha_i(y)]$ and $[y, \alpha_j(y)]$ are perpendicular. That is, within the span of X is a set of vertices of an $(n-1)$-dimensional cube where the $\mathbb{Z}_2^{n-1}$ action is transitive on vertices and the generators $\alpha_1, \ldots, \alpha_{n-1}$ act as reflections transposing distinct pairs of $(n-2)$-dimensional faces of the cube.

Let X be a set of points whose isometry group is (isomorphic to) $\mathbb{Z}_2^n$. Since the isometry group of X contains $\mathbb{Z}_2^{n-1}$, by the induction hypothesis, the span of X contains a point y as described above. Denote the $\mathbb{Z}_2^{n-1}$ orbit of y by Y. The proof divides into two cases depending on whether or not $\alpha_n(Y) \cap Y$ is empty.

Case 1. $\alpha_n(Y) \cap Y = \emptyset$.

Within $\alpha_n(Y) \cup Y$ we wish to find n points which we will label z_i so that each generator α_j fixes all z_i where $j \neq i$ and does not fix z_j. Given these points it will be easy to show that the line segments $\{[z_i, \alpha_i(z_i)]\}_{i=1}^n$ share a common midpoint and are mutually perpendicular. The convex hull of these points is then an (irregular) n-dimensional octahedron whose dual is the n-dimensional cube we need for our proof.

For each $i \leq n$ let z_i be the barycenter of the orbit of y under the subgroup generated by $\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$. Then if $i \neq j$, a straightforward computation shows that $\alpha_j(z_i) = z_i$. Our concern is that α_i might also fix z_i. Since no generator fixes y, we can show that α_i cannot fix both z_i and $z_i - \frac{y}{2} + \frac{\alpha_i(y)}{2}$. Further, for $i \neq j$, α_j fixes both z_i and $z_i - \frac{y}{2} + \frac{\alpha_i(y)}{2}$. Thus if α_i fixes z_i replace z_i with $z_i - \frac{y}{2} + \frac{\alpha_i(y)}{2}$. Thus we can find the z_i's we desire.

Since each z_i is the barycenter of exactly half of the $\mathbb{Z}_2^n$-images of y and $\alpha_i(z_i)$ is the barycenter of the other half, for each i the midpoint of $[z_i, \alpha_i(z_i)]$, $\frac{z_i + \alpha_i(z_i)}{2}$, is the barycenter of the $\mathbb{Z}_2^n$-orbit of y. Thus the line segments $\{[z_i, \alpha_i(z_i)]\}_{i=1}^n$ share a common midpoint.

Since $\alpha_j[z_j, z_i] = [\alpha_j(z_j), z_i]$, we see that z_i is equidistant from z_j and $\alpha_j(z_j)$. Similarly $\alpha_i(z_i)$ is equidistant from z_j and $\alpha_j(z_j)$. Using dot products of vectors to express length we have

$$\langle \vec{z}_i - \vec{z}_j, \vec{z}_i - \vec{z}_j \rangle = \langle \vec{z}_i - \alpha_j(\vec{z}_j), \vec{z}_i - \alpha_j(\vec{z}_j) \rangle.$$

Then

$$\|\vec{z}_i\|^2 - 2\langle \vec{z}_i, \vec{z}_j\rangle + \|\vec{z}_j\|^2 = \|\vec{z}_i^2\| - 2\langle \vec{z}_i, \alpha_j(\vec{z}_j)\rangle + \|\alpha_j(\vec{z}_j)\|^2 \Longrightarrow$$

$$2\langle \vec{z}_i, \vec{z}_j\rangle - \|\vec{z}_j\|^2 = 2\langle \vec{z}_i, \alpha_j(\vec{z}_j)\rangle - \|\alpha_j(\vec{z}_j)\|^2.$$

Similarly

$$2\langle \alpha_i(\vec{z}_i), \vec{z}_j\rangle - \|\vec{z}_j\|^2 = 2\langle \alpha_i(\vec{z}_i), \alpha_j(\vec{z}_j)\rangle - \|\alpha_j(\vec{z}_j)\|^2.$$

Combining the second with the first yields

$$\langle \vec{z}_i, \vec{z}_j\rangle - \langle \alpha_i(\vec{z}_i), \vec{z}_j\rangle = \langle \vec{z}_i, \alpha_j(\vec{z}_j)\rangle - \langle \alpha_i(\vec{z}_i), \alpha_j(\vec{z}_j)\rangle.$$

Thus $\langle \vec{z}_i - \alpha_i(\vec{z}_i), \vec{z}_j\rangle = \langle \vec{z}_i - \alpha_i(\vec{z}_i), \alpha_j(\vec{z}_j)\rangle$ or $\langle \vec{z}_i - \alpha_i(\vec{z}_i), \vec{z}_j - \alpha_j(\vec{z}_j)\rangle = 0$. Thus for each $i \neq j$, the vectors $\vec{z}_i - \alpha_j(\vec{z}_i)$ and $\vec{z}_j - \alpha_i(\vec{z}_j)$ are perpendicular. Further since the corresponding line segments share a common midpoint, the segments themselves are perpendicular.

We may choose $2n$ points on these line segments (within the span of X) that are equidistant from the common midpoint. The convex hull of these $2n$ points is an n-dimensional octahedron, the dual to the n-dimensional cube. There is a transitive isometric action of $\mathbb{Z}_2^n$ on the faces of the octahedron which gives a transitive $\mathbb{Z}_2^n$-action on the vertices of the dual cube. Further we can check that the generators $\alpha_1, \ldots, \alpha_n$ act as a reflections, each transposing opposite $(n-1)$-dimensional faces of the cube. Thus we have found the cube necessary for our induction argument.

Case 2. $\alpha_n(Y) \cap Y \neq \emptyset$.

Then there is some $\beta \in \mathbb{Z}_2^{n-1}$ so that $\alpha_n(y) = \beta(y) \Longrightarrow \beta\alpha_n(y) = y$. Replace α_n by $\beta\alpha_n$ as our n^{th} generator. Thus we may assume that α_n fixes y itself and therefore fixes Y pointwise.

Since, by the inductive hypothesis, Y is an $(n-1)$-dimensional cube with opposite faces transposed by the generators $\alpha_1, \ldots, \alpha_{n-1}$, we can take the dual of this cube and get an $(n-1)$-dimensional octahedron. For each pair of opposing $(n-2)$-dimensional faces of this cube, identify the generator α_i that transposes them and label the barycenter of one of these faces as z_i. The barycenter of the other is then $\alpha_i(z_i)$. The set of points $\{z_i, \alpha_i(z_i)\}_{i=1}^{n-1}$ is the vertex set of the octahedron that is dual to our $(n-1)$-dimension cube. Thus the line segments $\{[z_i, \alpha_i(z_i)]\}_{i=1}^{n-1}$ are mutually orthogonal and share a common midpoint – the barycenter of Y. Call this barycenter b and note that it is fixed by each of $\alpha_1, \ldots, \alpha_{n-1}$.

Since $\mathbb{Z}_2^n$ is the isometry group of X, there is some point in X that is moved by α_n. Call this point z_n. Suppose that the midpoint of $[z_n, \alpha_n(z_n)]$ is distinct from the common midpoint shared by each of $\{[z_i, \alpha_i(z_i)]\}_{i=1}^{n-1}$. Then we will "shift" the segment $[z_n, \alpha_n(z_n)]$ by $b - \frac{z_n+\alpha_n(z_n)}{2}$ so that these midpoints will be equal. That is, replace z_n with the point $z_n+b-\frac{z_n+\alpha_n(z_n)}{2}$. A simple computation shows that the midpoint of the new line segment $[z_n, \alpha_n(z_n)]$ is b.

Notice that since α_n fixes each point of Y, for each i both z_i and $\alpha_i(z_i)$ are equidistant from z_n and $\alpha_n(z_n)$. Using a computation identical to that of Case 1, it is easy to see that for each $1 \leq i \leq n-1$, the vectors $\vec{z_n} - \alpha_n(\vec{z_n})$ and $\vec{z_i} - \alpha_i(\vec{z_i})$ are perpendicular. Thus we have n mutually orthogonal line segments $\{[z_i, \alpha_i(z_i)]\}_{i=1}^{n}$ meeting at a common midpoint.

We now have exactly the same situation we had in Case 1. Therefore in this case also we can find the n-dimensional cube required by the inductive argument.

$\square$

3 Transitive Realizations

The last example from the introduction shows that the orbit number of $\mathbb{Z}_2^n$ is 1. We now show that no other abelian group of isometries can have a realization with just one orbit.

Theorem 2. If G is abelian and the orbit number of G equals 1, then $G \cong \mathbb{Z}_2^n$.

Proof. Suppose W is a one orbit realization of G. Since the stabilizers within an orbit are conjugate and G is abelian, the stabilizers are identical. Since W realizes G, the stabilizers are therefore trivial. Thus we can identify the points of W with the elements of G.

Consider $\gamma : W \to W$ given by $\gamma(w) = w^{-1}$. If $u, v \in W$,

$$dist(u, v) = dist((u^{-1}v^{-1})u, (u^{-1}v^{-1})v) =$$

$$dist(v^{-1}, u^{-1}) = dist(u^{-1}, v^{-1}) = dist(\gamma(u), \gamma(v)).$$

Thus γ is an isometry. It clearly fixes the identity element – but all stabilizers here are trivial. Thus γ must be the identity isometry. So if $w \in G$, then $w = w^{-1}$. Thus $G \cong \mathbb{Z}_2^n$.

$\square$

We close with some constructions of transitive realizations. There appear to be a wealth of these.

Examples:

1. S_3 in $\mathbb{R}^2$: Let $W = \left\{(1,0)\,,\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)\right\}$.

2. $S_3 \times \mathbb{Z}_2$ in $\mathbb{R}^3$:
   Let $W = \left\{(1,0,0)\,,\left(-\frac{1}{2},\frac{\sqrt{3}}{2},0\right),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2},0\right),\right.$
   $\left.(1,0,5)\,,\left(-\frac{1}{2},\frac{\sqrt{3}}{2},5\right),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2},5\right)\right\}$.

3. $S_3 \times S_3 \times \mathbb{Z}_2$ in $\mathbb{R}^4$:
   Let $W = \left\{(1,0,0,0)\,,\left(-\frac{1}{2},\frac{\sqrt{3}}{2},0,0\right),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2},0,0\right),\right.$
   $\left.(0,0,1,0)\,,\left(0,0,-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(0,0,-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)\right\}$.

4. $S_3 \times S_3$ in $\mathbb{R}^4$:
   Let $W = \left\{(1,0,0,0)\,,\left(-\frac{1}{2},\frac{\sqrt{3}}{2},0,0\right),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2},0,0\right),\right.$
   $\left.(0,0,2,0)\,,(0,0,-1,\sqrt{3})\,,(0,0,-1,-\sqrt{3})\right\}$.

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