Determining Number and Cost of Generalized Mycielskian Graphs

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Abstract

A set S of vertices is a *determining set* for a graph G if every automorphism of G is uniquely determined by its action on S. The size of a smallest determining set for G is called its *determining number*, $\mathrm{Det}(G)$. A graph G is said to be *d-distinguishable* if there is a coloring of the vertices with d colors so that only the trivial automorphism preserves the color classes. The smallest d for which G is d-distinguishable is its *distinguishing number*, $\mathrm{Dist}(G)$. If $\mathrm{Dist}(G) = 2$, the *cost of 2-distinguishing*, $\rho(G)$, is the size of a smallest color class over all 2-distinguishing colorings of G. The *Mycielskian*, $\mu(G)$, of a graph G is constructed by adding a shadow master vertex w, and for each vertex v_i of G adding a shadow vertex u_i, with edges so that the neighborhood of u_i in $\mu(G)$ is the same as the neighborhood of v_i in G with the addition of w. That is, $N(u_i) = N_G(v_i) \cup \{w\}$. The *generalized Mycielskian* $\mu^{(t)}(G)$ of a graph G is a Mycielskian graph with t layers of shadow vertices, each with edges to layers above and below, and the shadow master only adjacent to the top layer of shadow vertices. A graph is *twin-free* if it has no pair of vertices with the same set of neighbors. This paper examines the determining number and, when relevant, the cost of 2-distinguishing for Mycielskians and generalized Mycielskians of simple graphs with no isolated vertices. In particular, if $G \neq K_2$ is twin-free with no isolated vertices, then $\mathrm{Det}(\mu^{(t)}(G)) = \mathrm{Det}(G)$. Further, if $\mathrm{Det}(G) = k \geq 2$ and $t \geq k-1$, then $\mathrm{Dist}(\mu^{(t)}(G)) = 2$, and $\mathrm{Det}(\mu^{(t)}(G)) = \rho(\mu^{(t)}(G)) = k$. For G with twins, we develop a framework using quotient graphs with respect to equivalence classes of twin vertices to give bounds on the determining number of Mycielskians. Moreover, we identify classes of graphs with twins for which $\mathrm{Det}(\mu^{(t)}(G)) = (t{+}1)\,\mathrm{Det}(G)$.

1 Introduction

In 1955, Mycielski introduced a construction [19] that takes a finite simple graph, G, and produces a larger graph, $\mu(G)$, called the (*traditional*) *Mycielskian* of G, with a strictly larger chromatic number. To construct $\mu(G)$, begin with a copy of G. For each $v \in V(G)$, add a *shadow vertex* u and edges between u and the neighbors of v. Finally, add a vertex w whose neighbors are precisely the shadow vertices. We call w the *shadow master*. A more formal definition can be found in Section 2.

Mycielski iteratively applied this construction to $G = K_2$, creating what are now called the *classic Mycielski graphs*, M_n. More precisely, $M_2 = K_2$ and for all $k \geq 1$, $M_{k+2} = \mu(M_{k+1}) = \mu_k(K_2)$, where we use $\mu_k(G)$ to indicate iteratively applying the Mycielsian construction k times starting with G. Thus, $M_3 = \mu(K_2) = C_5$, and $M_4 = \mu(M_3) = \mu_2(K_2)$ is commonly called the Grötsch graph. Mycielski proved that these graphs are all triangle-free and satisfy $\chi(M_n) \geq n$, where $\chi(G)$ is the chromatic number of G.

The *generalized Mycielskian* of graph G was defined by Stiebitz [21] in 1985 (cited in [22]) and independently by Van Ngoc [23] in 1987 (cited in [24]). It is denoted $\mu^{(t)}(G)$ and will also be formally defined in Section 2. This construction can be described as having a copy of G at level 0 and $t \geq 1$ levels of shadow vertices whose neighborhoods extend to the levels above and below and that mirror the neighborhoods of the vertices of G. Finally, $\mu^{(t)}(G)$ has a shadow master w that is adjacent to each shadow vertex at the top level, t. Note that $\mu^{(1)}(G) = \mu(G)$. The generalized Mycielski construction is used to construct graphs with arbitrarily large odd girth and arbitrarily large chromatic number.

In this paper, we will compare the determining number and, when relevant, the cost of 2-distinguishing for a finite simple graph G to the same parameters for the Mycielskian graphs arising from G. These parameters are defined and motivated below.

A coloring of the vertices of a graph G with the colors $1, \ldots, d$ is called a *d-distinguishing coloring* if no nontrivial automorphism of G preserves the color classes. A graph is called *d-distinguishable* if it has a d-distinguishing coloring. The distinguishing number of G, denoted $\text{Dist}(G)$, is the smallest number of colors necessary for a distinguishing coloring of G. Albertson and Collins introduced graph distinguishing in [3]. Most of the work in graph distinguishing in the last few decades has proved that, for a large number of graph families, all but a finite number of members are 2-distinguishable. Examples of such families of finite graphs include: hypercubes Q_n with $n \geq 4$ [6], Cartesian powers G^n for a connected graph $G \neq K_2, K_3$ and $n \geq 2$ [1, 15, 17], and Kneser graphs $K_{n:k}$ with $n \geq 6, k \geq 2$ [2]. Examples of such families of infinite graphs include: the denumerable random graph [16], the infinite hypercube [16], and denumerable vertex-transitive graphs of connectivity 1 [20].

In 2007, Imrich [14] asked whether distinguishing could be refined to provide more information within the class of 2-distinguishable graphs. In response, Boutin [10] defined the *cost of 2-distinguishing* a 2-distinguishable graph G to be the minimum size of a color class over all 2-distinguishing colorings of G.

The cost of 2-distinguishing G is denoted $\rho(G)$.

Some of the graph families with known or bounded cost are hypercubes with $\lceil \log_2 n \rceil + 1 \leq \rho(Q_n) \leq 2\lceil \log_2 n \rceil - 1$ for $n \geq 5$ [10], Kneser graphs with $\rho(K_{2^m-1:2^{m-1}-1}) = m+1$ [13], and $\rho(K_{2^m} \square H) = m \cdot 2^{m-1}$, where $\square$ denotes the Cartesian product and H is a graph with no nontrivial automorphisms [8].

A determining set is a useful tool in finding the distinguishing number and, when relevant, the cost of 2-distinguishing. A subset $S \subseteq V(G)$ is said to be a *determining set* for G if the only automorphism that fixes the elements of S pointwise is the trivial automorphism. Equivalently, S is a determining set for G if whenever φ and ψ are automorphisms of G with $\varphi(x) = \psi(x)$ for all $x \in S$ then $\varphi = \psi$ [9]. The *determining number* of a graph G, denoted $\mathrm{Det}(G)$, is the size of a smallest determining set. Intuitively, if we think of automorphisms of a graph as allowing vertices to move among their relative positions, one can think of the determining number as the smallest number of pins needed to "pin down" the graph.

For some graph families, we only have bounds on the determining number. For instance, for Kneser graphs, $\log_2(n+1) \leq \mathrm{Det}(K_{n:k}) \leq n-k$ with both upper and lower bounds sharp [9]. However, there are families for which we know the determining numbers of its members exactly. In particular, for hypercubes, $\mathrm{Det}(Q_n) = \lceil \log_2 n \rceil + 1$, and for Cartesian powers $\mathrm{Det}(K_3^n) = \lceil \log_3(2n+1) \rceil + 1$ [11].

Though distinguishing numbers and determining numbers were introduced by different people and for different purposes, they have strong connections. Albertson and Boutin showed in [2] that if G has a determining set S of size d, then giving each vertex in S a distinct color from $1, \ldots, d$ and every other vertex color $d+1$ gives a $(d+1)$-distinguishing coloring of G. Thus, $\mathrm{Dist}(G) \leq \mathrm{Det}(G)+1$.

Further, in [12], Boutin pointed out that, given a 2-distinguishing coloring of G, since only the trivial automorphism preserves the color classes setwise, only the trivial automorphism preserves them pointwise. Consequently, each of the color classes in a 2-distinguishing coloring is a determining set for the graph, though not necessarily of minimum size. Thus, if G is 2-distinguishable, then $\mathrm{Det}(G) \leq \rho(G)$.

In 2018, Alikhani and Soltani [4] studied the distinguishing number of the traditional Mycielskian of a graph. In particular, they showed that the classic Mycielski graphs $M_{k+2} = \mu_k(K_2)$ satisfy $\mathrm{Dist}(M_n) = 2$ for all $n \geq 4$. To generalize to Mycielskians of arbitrary graphs, they considered the role of twin vertices. Two vertices in a graph are said to be *twins* if they have the same open neighborhood, and a graph is said to be *twin-free* if it does not contain any twins. In particular, Alikhani and Soltani proved that if G is twin-free with at least two vertices, then $\mathrm{Dist}(\mu(G)) \leq \mathrm{Dist}(G)+1$. Further, they conjectured that for all but a finite number of connected graphs G with at least 3 vertices, $\mathrm{Dist}(\mu(G)) \leq \mathrm{Dist}(G)$. In [7], Boutin, Cockburn, Keough, Loeb, Perry, and Rombach proved the conjecture with the theorem stated below. Notice that this theorem does not require graph connectedness.

Theorem 1.1. *[7] Let $G \neq K_1, K_2$ be a graph with $\ell \geq 0$ isolated vertices. If $t\ell > \mathrm{Dist}(G)$, then $\mathrm{Dist}(\mu^{(t)}(G)) = t\ell$. Otherwise, $\mathrm{Dist}(\mu^{(t)}(G)) \leq \mathrm{Dist}(G)$.*

As seen in Theorem 1.1 and [18, 5], the presence of isolated vertices in G, and therefore in $\mu^{(t)}(G)$, has a significant effect on the structure and behavior of $\mu^{(t)}(G)$. If v_i is an isolated vertex in G and $t \geq 2$, then in $\mu^{(t)}(G)$, $v_i = u_i^0, u_i^1, \ldots u_i^{t-1}$ are all isolated vertices and, hence, mutually twins. Thus, it is common to exclude isolated vertices when studying the Mycielski constructions. In this paper, we will restrict our attention to finite simple graphs that are not necessarily connected, but that have no isolated vertices. Analogous results for Mycielskians of graphs with isolated vertices, including K_1, will be covered in a forthcoming paper.

In Section 2, the formal definitions of the traditional and generalized Mycielskians of a graph, and lemmas regarding the action of their automorphisms are developed. The twin-free case is considered in Section 3. In particular, we prove that if a graph G is not K_2 and is twin-free with no isolated vertices, then

$$\mathrm{Det}(\mu^{(t)}(G)) = \mathrm{Det}(G).$$

We also show that if, in addition to the above hypotheses, G has determining number $k \geq 2$ and $t \geq k-1$, then

$$\mathrm{Dist}(\mu^{(t)}(G)) = 2 \quad \text{and} \quad \rho(\mu^{(t)}(G)) = \mathrm{Det}(G).$$

Finally, in Section 4, we prove results on the determining number for the Mycielski construction applied to graphs with twins. To accomplish this we develop a technique utilizing equivalence classes of twins and the resulting quotient graph. This technique allows us to show that the presence of twin vertices in G causes the determining number of $\mu^{(t)}(G)$ to grow proportionally with the number of vertices. More precisely, if G is a graph with twins and has a determining set consisting only of twins, then

$$\mathrm{Det}(\mu^{(t)}(G)) = (t+1)\,\mathrm{Det}(G).$$

Additionally, if G does not have a determining set consisting only of twins, we give bounds on the determining number of $\mu^{(t)}(G)$.

2 Generalized Mycielskian Graphs

In this section we formally define the traditional and generalized Mycielskians of a graph, and we present observations about twin vertices in these graphs. We then discuss how automorphisms of the Mycielskian graph behave when the underlying graph is twin-free and has no isolated vertices.

Throughout this paper, let $N_G(v)$ be the open neighborhood of v in G. For ease of notation, the open neighborhood of v in the Mycielskian graph will simply be denoted $N(v)$.

Suppose G is a finite simple graph with $V(G) = \{v_1, \ldots, v_n\}$. The *Mycielskian of* G, denoted $\mu(G)$, is a graph with vertex set

$$V(\mu(G)) = \{v_1, \ldots, v_n, u_1, \ldots u_n, w\}.$$

For each edge $v_iv_j \in E(G)$, $\mu(G)$ has edges v_iv_j, u_iv_j and v_iu_j; and additionally, u_iw for all $1 \leq i \leq n$. That is, $N(u_i) = N_G(v_i) \cup \{w\}$ and $\mu(G)$ contains G as an induced subgraph on the vertex set $\{v_1, \ldots, v_n\}$. We refer to the vertices $v_1, \ldots, v_n$ in $\mu(G)$ as *original vertices* and the vertices $u_1, \ldots, u_n$ as *shadow vertices*. Since w dominates the shadow vertices, we refer to it as the *shadow master*. Further, notice that the shadow vertices form an independent set. See Figure 1 for $\mu(K_2)$ and $\mu(K_3)$.

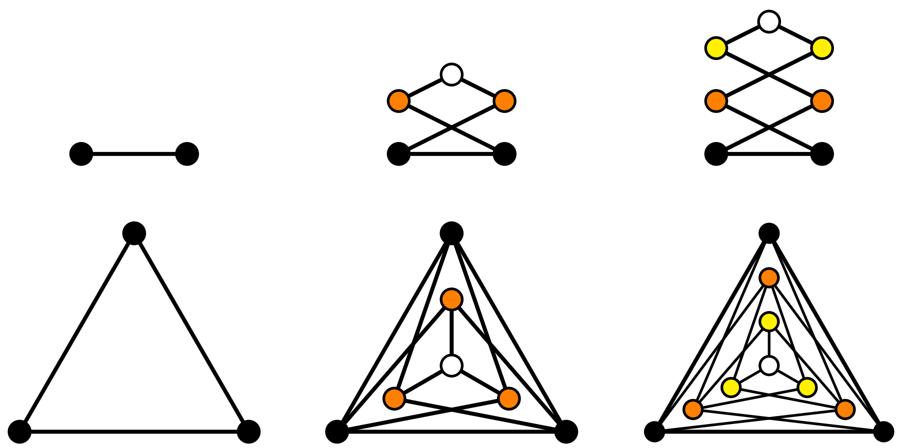

Figure 1: Top: K_2, $\mu(K_2)$ and $\mu^{(2)}(K_2)$, drawn with vertical levels with the shadow master on the top. Bottom: K_3, $\mu(K_3)$ and $\mu^{(2)}(K_3)$, drawn with concentric levels with the shadow master in the middle.

We now establish an observation about the relationship between automorphisms of graphs and automorphisms of their Mycielskians.

Observation 2.1. *An automorphism α of G induces an automorphism $\widehat{\alpha}$ on $\mu(G)$ by replicating the action of α on the set of original vertices and also on the set of shadow vertices, and leaving the shadow master fixed. That is, we can define the automorphism $\widehat{\alpha}$ of $\mu(G)$ by $\widehat{\alpha}(v_i) = \alpha(v_i) = v_j$ and $\widehat{\alpha}(u_i) = u_j$ and $\widehat{\alpha}(w) = w$.*

We will see that for most graphs, the only automorphisms of $\mu(G)$ are those induced by automorphisms of G. Making this more precise will involve a discussion of twin vertices. Recall from the introduction that two vertices x and y are called twins if they have precisely the same set of neighbors. It is possible to have a collection of three or more mutually twin vertices. If two vertices of G are twins, it is straightforward to show that exchanging the twins and fixing the remaining vertices is an an automorphism of G. Furthermore, also recall from the introduction that a set S is a determining set for G if the only automorphism that fixes the elements of S pointwise is the trivial automorphism. Thus, every determining set must contain all but one representative from every collection of mutually twin vertices.

The following are some structural observations about relationships between twins in G and twins in $\mu(G)$.

Observation 2.2. *In $\mu(G)$, the shadow master w is adjacent to each shadow vertex, but not adjacent to any original vertex, so twins in $\mu(G)$ are either both shadow vertices or both original vertices.*

Observation 2.3. *If v_i and v_j are twins in G, then they are twins in $\mu(G)$ and so are their shadows u_i and u_j.*

Observation 2.4. *If at least one pair of $\{v_i, v_j\}$ or $\{u_i, u_j\}$ is twins in $\mu(G)$, then both pairs are, as is $\{v_i, v_j\}$ within G.*

In [4], Alikhani and Soltani considered automorphisms of $\mu(G)$ when G is twin-free, and proved the following.

Lemma 2.5. *[4] If G is twin-free and $\widehat{\alpha}$ is an automorphism of $\mu(G)$ that fixes the shadow master, then*

(i) $\widehat{\alpha}$ preserves the set of original vertices, $\{v_1, \ldots, v_n\}$, and the set of shadow vertices, $\{u_1, \ldots, u_n\}$;

(ii) $\widehat{\alpha}$ restricted to $\{v_1, \ldots, v_n\}$ is an automorphism α of G;

(iii) $\alpha(v_i) = v_j$ if and only if $\widehat{\alpha}(u_i) = u_j$.

Thus, if G is twin-free, then every automorphism of $\mu(G)$ that fixes the shadow master is induced by an automorphism of G.

The *generalized Mycielskian* of G, also known as a *cone over* G, was defined by Stiebitz [21] in 1985 (cited in [22]) and independently by Van Ngoc [23] in 1987 (cited in [24]), and has multiple levels of shadow vertices. More precisely, for $t \in \mathbb{N}$, the generalized Mycielskian of G, denoted $\mu^{(t)}(G)$, has vertex set

$$\{u_1^0, \ldots, u_n^0, u_1^1, \ldots, u_n^1, \ldots, u_1^t, \ldots, u_n^t, w\}.$$

For each edge $v_i v_j$ in G, the graph $\mu^{(t)}(G)$ has edge $v_i v_j = u_i^0 u_j^0$, as well as edges $u_i^s u_j^{s+1}$ and $u_j^s u_i^{s+1}$, for $0 \leq s < t$. Finally, $\mu^{(t)}(G)$ has edges $u_i^t w$ for $1 \leq i \leq n$. Intuitively, for $1 \leq s \leq t-1$, the neighbors of the shadow vertex u_i^s are the shadows of the neighbors of v_i both at level $s-1$ and at level $s+1$, while the neighbors of u_i^t are the shadows of the neighbors of v_i at level $t-1$ and the shadow master w.

We say that vertex u_i^s is *at level* s. In addition, we make the identification $u_i^0 = v_i$. As we did for the traditional Mycielskian, we refer to the vertices at level 0 as *original vertices*, to the vertices at level $1 \leq s \leq t$ as *shadow vertices*, and to w as the *shadow master*. Since the shadow master dominates only the shadow vertices at level t, we call level t the *top level*. Since $\mu^{(1)}(G) = \mu(G)$, we omit the superscripts when $t = 1$. In Figure 1, we illustrate both the traditional Mycielskian and the generalized Mycielskian with $t = 2$ for graphs K_2 and K_3.

As one might suspect, Observations 2.1, 2.2, 2.3, and 2.4, as well as Lemma 2.5, for traditional Mycielskian graphs extend to the generalized Mycielskians with only minor changes.

Observation 2.6. *An automorphism α of G induces an automorphism $\widehat{\alpha}$ on $\mu^{(t)}(G)$ by replicating the action of α on each level of $\mu^{(t)}(G)$. That is, we can define the automorphism $\widehat{\alpha}$ of $\mu^{(t)}(G)$ by $\widehat{\alpha}(u_i^0) = \alpha(v_i) = v_j$ and $\widehat{\alpha}(u_i^s) = u_j^s$ for each $1 \leq s \leq t$ and $\widehat{\alpha}(w) = w$.*

Observation 2.7. *If G has no isolated vertices, twin vertices in $\mu^{(t)}(G)$ must be vertices at the same level. That is, twin vertices have the form u_i^s and u_j^s for some $0 \leq s \leq t$.*

Observation 2.8. *If original vertices v_i and v_j are twins in G, then u_i^s and u_j^s are twins in $\mu^{(t)}(G)$ for all $0 \leq s \leq t$.*

Observation 2.9. *If $\{u_i^s, u_j^s\}$ are twins in $\mu^{(t)}(G)$ for any $0 \leq s \leq t$, then they are twins for all such s. In particular, $\{v_i, v_j\}$ are twins in G.*

Boutin, Cockburn, Keough, Loeb, Perry and Rombach prove in [7] that if G is not $K_{1,m}$ for $m \geq 0$, then all automorphisms of $\mu^{(t)}(G)$ fix the shadow master.

Lemma 2.10. *[7] Let G be a graph and let $t \in \mathbb{N}$. Let $\widehat{\alpha}$ be an automorphism of $\mu^{(t)}(G)$.*

(i) If $G = K_{1,1} = K_2$, then $\mu^{(t)}(G) = C_{2t+3}$, and $\widehat{\alpha}(w)$ can be any vertex.

(ii) If $G = K_{1,m}$ with $m \neq 1$ and v is the vertex of degree m in G, then $\widehat{\alpha}(w) \in \{w, u^t\}$, where u^t is the top level shadow of v.

(iii) If $G \neq K_{1,m}$ for any $m \geq 0$, then $\widehat{\alpha}(w) = w$.

In essence, Lemma 2.10 states that the only graphs G that have automorphisms of $\mu^{(t)}(G)$ that do not fix w are the star graphs $K_{1,m}$. Figure 2 shows the graphs $K_{1,3}$ and $\mu^{(t)}(K_{1,3})$ for $t = 1, 2, 3$. The vertical reflectional symmetries in the drawings of $\mu^{(t)}(K_{1,3})$ in Figure 2 correspond to automorphisms that move the shadow master to the top level shadow of the central vertex in $K_{1,3}$.

The next lemma is an extension of Lemma 2.5 to generalized Mycielskian graphs.

Lemma 2.11. *Let $G \neq K_{1,m}$ for any $m \geq 0$ be a graph on n vertices that has no isolated vertices and let $t \in \mathbb{N}$. If $\widehat{\alpha}$ is an automorphism of $\mu^{(t)}(G)$, then*

(i) $\widehat{\alpha}$ preserves the level of vertices, that is, $\widehat{\alpha}(\{u_1^s \dots, u_n^s\}) \subseteq \{u_1^s \dots, u_n^s\}$ for all $0 \leq s \leq t$;

(ii) $\widehat{\alpha}$ restricted to $\{u_1^0, \dots, u_n^0\} = \{v_1, \dots, v_n\}$ is an automorphism α of G;

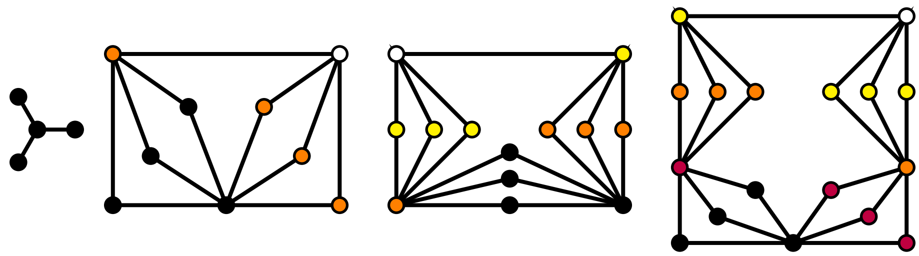

Figure 2: The graphs $K_{1,3}$ and $\mu^{(t)}(K_{1,3})$ for $t = 1, 2, 3$. In each, the vertices of matching color are at the same level.

(iii) if, in addition, G is twin-free, then $\alpha(v_i) = v_j$ if and only if $\widehat{\alpha}(u_i^s) = u_j^s$ for all $0 < s \leq t$.

Proof. Since $G \neq K_{1,m}$ for any $m \geq 0$, by Lemma 2.10(iii), any automorphism of $\mu^{(t)}(G)$ fixes the shadow master. Furthermore, since G has no isolated vertices, every vertex x in $\mu^{(t)}(G)$ can be uniquely determined by its distance from w. That is, in $\mu^{(t)}(G)$, $x = w$ if and only if the distance of x from w is 0, and x is at level s if and only if its distance from w is $t{+}1{-}s$. Since distance is preserved by automorphisms, any automorphism that fixes the shadow master preserves the level of every other vertex. This proves (i) and (ii).

The proof of Lemma 2.5(iii) that was given by Alikhani and Soltani in [4] extends to statement (iii) here in the following way. Focusing first on α, if $\alpha(v_i) = v_j$, then α maps the open neighborhood of v_i to the open neighborhood of v_j. That is, $\alpha(v_i) = v_j$ implies $\alpha(N_G(v_i)) = N_G(v_j)$. Further, since G is twin-free, every vertex in G can be uniquely identified by its open neighborhood. Thus, $\alpha(N_G(v_i)) = N_G(v_j)$ implies $\alpha(v_i) = v_j$. This gives us the biconditional statement $\alpha(N_G(v_i)) = N_G(v_j)$ if and only if $\alpha(v_i) = v_j$.

Since G has no isolated vertices and is twin-free, Observations 2.7 and 2.9 imply that $\mu^{(t)}(G)$ is also twin-free, so each vertex of $\mu^{(t)}(G)$ can be uniquely identified by its open neighborhood. Hence, $\widehat{\alpha}(u_i^s) = u_j^s$ if and only if $\widehat{\alpha}(N(u_i^s)) = N(u_j^s)$. By definition,

$$N(u_i^s) = \begin{cases} \{u_k^0, u_k^1 \mid v_k \in N_G(v_i)\}, & \text{if } s = 0, \\ \{u_k^{s-1}, u_k^{s+1} \mid v_k \in N_G(v_i)\}, & \text{if } 0 < s < t, \\ \{u_k^{t-1} \mid v_k \in N_G(v_i)\} \cup \{w\}, & \text{if } s = t. \end{cases}$$

Thus, the open neighborhood of u_i^s in $\mu^{(t)}(G)$ is completely determined by the open neighborhood of $u_i^0 = v_i$ in G. This, in turn, implies that $\widehat{\alpha}(N(u_i^s)) = N(u_j^s)$ if and only if $\alpha(N_G(v_i)) = N_G(v_j)$. Together with the biconditional statement from the previous paragraph (iii) follows. $\square$

3 $\mathrm{Det}(\mu^{(t)}(G))$ and $\rho(\mu^{(t)}(G))$ for Twin-Free G

We will begin this section by proving that for most twin-free graphs the generalized Mycielski construction preserves determining number. This is in contrast to the proven effect of the Mycielski construction on distinguishing number. More explicitly, in [7], the current authors proved that for $G \neq K_1, K_2$, $\mathrm{Dist}(\mu(G)) \leq \mathrm{Dist}(G)$, and that these values may be arbitrarily far apart. For example, if $n \geq 3$ and $t \geq \log_2(n-1)$, then $\mathrm{Dist}(\mu^{(t)}(K_n)) = 2$, whereas $\mathrm{Dist}(K_n) = n$.

Theorem 3.1. *Let G be a twin-free graph with no isolated vertices and let $t \in \mathbb{N}$.*

(i) If $G = K_2$ then, $\mathrm{Det}(G) = 1$ *and* $\mathrm{Det}(\mu^{(t)}(G)) = 2$.

(ii) If $G \neq K_2$ then any minimum size determining set for G is a minimum size determining set for $\mu^{(t)}(G)$ and

$$\mathrm{Det}(\mu^{(t)}(G)) = \mathrm{Det}(G).$$

Proof. For (i), if $G = K_2$, then $\mathrm{Det}(G) = 1$ and, $\mathrm{Det}(\mu^{(t)}(G)) = 2$ since $\mu^{(t)}(K_2) = C_{2t+3}$.

For (ii), let $S \subseteq V(G)$ be a determining set for G; we will show S, as a subset of $V(\mu^{(t)}(G))$, is also a determining set for $\mu^{(t)}(G)$. Let $\widehat{\alpha} \in \mathrm{Aut}(\mu^{(t)}(G))$ such that $\widehat{\alpha}(s) = s$ for all $s \in S$. By Lemma 2.11(ii), the restriction of $\widehat{\alpha}$ to $V(G)$ is an automorphism $\alpha \in \mathrm{Aut}(G)$ and, further, by Lemma 2.11(iii), $\alpha(v_i) = v_j$ if and only if $\widehat{\alpha}(u_i^s) = u_j^s$ for all $0 \leq s \leq t$. So, by the assumption that $\widehat{\alpha}$ fixes S pointwise and that S is a determining set for G, α is the identity on the original vertices. Thus, $\widehat{\alpha}(v_i) = \alpha(v_i) = v_i$ for all $v_i \in V(G)$ and hence, $\widehat{\alpha}(u_i^s) = u_i^s$ for all i and all s. Thus, S is a determining set for $\mu^{(t)}(G)$ and so $\mathrm{Det}(\mu^{(t)}(G)) \leq \mathrm{Det}(G)$.

Now, suppose instead that $S \subseteq V(\mu^{(t)}(G))$ is a minimum size determining set for $\mu^{(t)}(G)$. Since G is twin-free, has no isolated vertices, and is not K_2, by Lemma 2.10(iii), every automorphism of $\mu^{(t)}(G)$ fixes the shadow master w and so, by the minimality of S, we can assume $w \notin S$. Let

$$S_0 = \{v_i \in V(G) \mid u_i^s \in S \text{ for some } 0 \leq s \leq t\}.$$

Then $|S_0| \leq |S|$. If $\beta \in \mathrm{Aut}(G)$ fixes S_0 pointwise, then by Lemma 2.11(iii), the automorphism $\widehat{\beta}$ on $\mu^{(t)}(G)$ fixes S. Thus, $\widehat{\beta}$ is the identity on $\mu^{(t)}(G)$ and so restricts to the identity on G. Hence, S_0 is a determining set for G and so $\mathrm{Det}(G) \leq \mathrm{Det}(\mu^{(t)}(G))$, yielding equality. □

Now, we consider graphs with known determining number. By definition, $\mathrm{Det}(G) = 0$ if and only if G has only the trivial automorphism, or, equivalently, if G is asymmetric. In particular, $\mathrm{Det}(G) = 0$ if and only if $\mathrm{Dist}(G) = 1$. If G has nontrivial automorphisms, then $\mathrm{Det}(G) = 1$ if and only if G has a vertex x that forms a singleton determining set. In this case, we can color x red and all other vertices blue to obtain a 2-distinguishing coloring of G. This coloring shows that if $\mathrm{Det}(G) = 1$, then $\mathrm{Dist}(G) = 2$ and $\rho(G) = 1$. Note that these facts hold for graphs with or without twins.

In fact, if $G \neq K_2$ is a twin-free graph with no isolated vertices and $\mathrm{Det}(G) = 1$, then by Theorem 3.1, $\mathrm{Det}(\mu^{(t)}(G)) = \mathrm{Det}(G) = 1$. Thus, since $\mathrm{Det}(\mu^{(t)}(G)) = 1$, we have $\mathrm{Dist}(\mu^{(t)}(G)) = 2$ and $\rho(\mu^{(t)}(G)) = 1$.

Theorem 3.2. *Let G be a twin-free graph with no isolated vertices such that $\mathrm{Det}(G) = k \geq 2$. Then for $t \geq \lceil \log_2(k+1) \rceil - 1$,*

$$\mathrm{Det}(\mu^{(t)}(G)) = k, \mathrm{Dist}(\mu^{(t)}(G)) = 2, \textit{ and } \rho(\mu^{(t)}(G)) \leq \frac{(k+1)\lceil \log_2(k+1) \rceil}{2}.$$

Proof. Since $\mathrm{Det}(K_2) = 1$ and we are assuming $\mathrm{Det}(G) \geq 2$ we have $G \neq K_2$. Thus, by Theorem 3.1, $\mathrm{Det}(\mu^{(t)}(G)) = \mathrm{Det}(G) = k$. Further, since G is twin-free and has no isolated vertices, $G \neq K_{1,m}$ for any $m \geq 0$.

To show $\mathrm{Dist}(\mu^{(t)}(G)) = 2$, first observe that since $\mathrm{Det}(\mu^{(t)}(G)) \geq 2$, $\mu^{(t)}(G)$ has a nontrivial automorphism and so cannot be distinguished with one color. Thus, $\mathrm{Dist}(\mu^{(t)}(G)) \geq 2$.

We will now show $\mathrm{Dist}(\mu^{(t)}(G)) \leq 2$. Let $r = \lceil \log_2(k+1) \rceil$ and let $S = \{v_1, v_2, \ldots, v_k\}$ be a determining set for G. For $1 \leq i \leq k$, let $b_1 b_2 \ldots b_r$ be the binary representation of i with leading zeros if necessary. For each $1 \leq i \leq k$ and each $0 \leq j \leq r-1$, color u_i^j red if $b_j = 1$ and u_i^j blue if $b_j = 0$. We color all other vertices blue. Assume $\widehat{\alpha} \in \mathrm{Aut}(\mu^{(t)}(G))$ preserves the red and blue color classes. By Lemma 2.11(i), since G is not a star graph, $\widehat{\alpha}$ preserves levels. Furthermore, by Lemma 2.11(iii) we know that $\widehat{\alpha}(v_i) = v_j$ if and only if $\widehat{\alpha}(u_i^s) = u_j^s$ for all $0 < s \leq t$. For every $u_i^0 = v_i \in S$, the distinct sequence of colors in the ordered set $\{u_i^0, \ldots, u_i^r\}$ guarantees that $\widehat{\alpha}(v_i) = v_i$ for all $1 \leq i \leq k$. By the fact that S is a determining set for G, and by Lemma 2.11(ii) and (iii), we now have that $\widehat{\alpha}$ fixes the vertices at level 0 and therefore fixes every vertex in $\mu^{(t)}(G)$. This shows that $\mathrm{Dist}(\mu^{(t)}(G)) \leq 2$ and completes the proof that $\mathrm{Dist}(\mu^{(t)}(G)) = 2$. Furthermore, this coloring has no more than $(k+1)\frac{r}{2} = (k+1)\lceil \log_2(k+1) \rceil / 2$ red vertices, which gives us the upper bound on $\rho(\mu^{(t)}(G))$. □

Note that the bound on t in Theorem 3.2 is sharp. To see this, consider $G = K_5$ and $t = 1$. Since $\mathrm{Det}(K_5) = 4$, we have $t < \lceil \log_2(4+1) \rceil - 1$. By Lemma 2.10(iii), every automorphism of $\mu(K_5)$ fixes w, and by Lemma 2.5(ii), the remaining vertices are mapped as pairs. That is, every pair (v_i, u_i) can be mapped to any other pair (v_j, u_j). Thus, in a 2-distinguishing coloring of $\mu(K_5)$, each of the five vertex pairs must have a distinct 2-coloring. However, since there are precisely four ways in which we can 2-color a pair (v_i, u_i), we see that in any 2-coloring of $\mu(K_5)$ at least two of the vertex pairs must have the same coloring. Thus, no 2-coloring of $\mu(K_5)$ can be distinguishing. So we see that the bound $t \geq \lceil \log_2(4+1) \rceil - 1$ is sharp.

The bound on $\rho(\mu^{(t)}(G))$ is also sharp. To see this, consider $G = K_4$ and $t = 1$. Since $\mathrm{Det}(K_4) = 3$ and $t < \lceil \log_2(3+1) \rceil - 1$, Theorem 3.2 applies.

To explicitly find $\rho(\mu(K_4))$, as in the previous example, note that in any 2-distinguishing coloring of $\mu(K_4)$ the ordered pairs of the form (v_i, u_i) must be distinguished from each other. This forces us to use all four of the ordered pairs of two colors. This implies that we must use each of the colors on at least 4 vertices. Therefore, we have $\rho(\mu(K_4)) = ((k+1)\lceil \log_2(k+1) \rceil)/2 = 4$, precisely our upper bound on $\rho(\mu(G))$.

For many values of t and k, it is possible to to find a coloring that does better in terms of the cost than the method above. For example, if $k+1$ is not a power of 2, we may choose a set of k integers in the range $1, \ldots, 2^m$ that minimizes the number of 1s in their binary representations. When $t > m-1$, then choosing k integers in the range $1, \ldots, 2^{t+1}$ gives a similar added flexibility. The next theorem makes that flexibility precise when $t \geq k-1$.

Theorem 3.3. *Let G be a twin-free graph with no isolated vertices such that* $\mathrm{Det}(G) = k \geq 2$. *Then for $t \geq k-1$,*

$$\mathrm{Det}(\mu^{(t)}(G)) = k, \mathrm{Dist}(\mu^{(t)}(G)) = 2, \text{ and } \rho(\mu^{(t)}(G)) = k.$$

Proof. Since $t \geq k-1$ we have $t \geq \lceil \log_2(k+1) \rceil - 1$. So by Theorem 3.2, $\mathrm{Det}(\mu^{(t)}(G)) = k$ and $\mathrm{Dist}(\mu^{(t)}(G)) = 2$.

Let $\widehat{S} = \{u_1^0, u_2^1, \ldots, u_k^{k-1}\}$. Color the vertices in $\widehat{S}$ red and all other vertices blue. The proof that this is a 2-distinguishing coloring of $\mu^{(t)}(G)$ is similar to the proof in Theorem 3.2.

Now, since $|\widehat{S}| = k$, the size of a color class in the 2-distinguishing coloring above, $\rho(\mu^{(t)}(G)) \leq k$. If there is a 2-distinguishing coloring of $\mu^{(t)}(G)$ with a color class of size $k-1$, then $\mu^{(t)}(G)$ would have a determining set of size $k-1$. Since $\mathrm{Det}(\mu^{(t)}(G)) = k$, we can now conclude that $\rho(\mu^{(t)}(G)) = k$ when $t \geq k-1$. □

As noted earlier, Alikhani and Soltani [4] showed that the classic Mycielski graphs $M_{k+2} = \mu_k(K_2)$ satisfy $\mathrm{Dist}(M_n) = 2$ for any $n \geq 4$. We can obtain this result and more by noting that $M_3 = C_5 \neq K_2$ is twin-free, has no isolated vertices, and satisfies $\mathrm{Det}(M_3) = \mathrm{Det}(C_5) = 2$. We now apply Theorem 3.3 iteratively, with t mercifully equal to 1, to achieve the following.

Corollary 3.4. *For all $n \geq 4$,* $\mathrm{Det}(M_n) = \mathrm{Dist}(M_n) = \rho(M_n) = 2$.

4 $\mathrm{Det}(\mu^{(t)}(G))$ for G with Twins

We next consider graphs with twin vertices. For vertices x, y of a graph G, define $x \sim y$ if x and y are twin vertices. It is easy to verify that $\sim$ is an equivalence relation on $V(G)$.

The quotient graph with respect to the relation $\sim$, denoted $\widetilde{G}$, has as its vertices the set of equivalence classes $[x] = \{y \in V(G) \mid x \sim y\}$ with $[x]$ adjacent to $[z]$ in $\widetilde{G}$ if and only if there exist $p \in [x]$ and $q \in [z]$ such that p and q are adjacent in G. By definition of $\sim$, all vertices in an equivalence class have

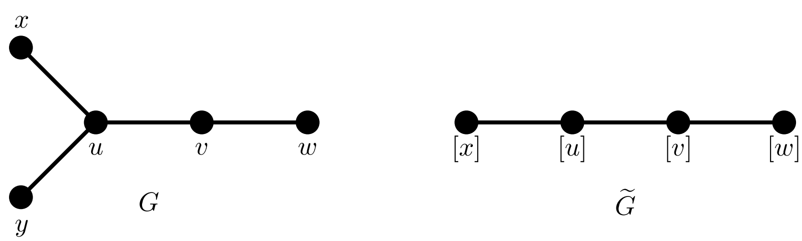

Figure 3: A graph G and its quotient graph $\widetilde{G}$.

the same neighbors, so in our case $[x]$ is adjacent to $[z]$ in $\widetilde{G}$ if and only if x is adjacent to z in G. Thus,

$$N_{\widetilde{G}}([x]) = \{[z] \mid z \in N_G(x)\}.$$

In particular, if $N_{\widetilde{G}}([x]) = N_{\widetilde{G}}([y])$ then $N_G(x) = N_G(y)$ and so $[x] = [y]$. This implies that $\widetilde{G}$ is twin-free. In fact, G is twin-free if and only if $\widetilde{G} = G$. In this section we focus on graphs that have twins, that is, graphs for which $\widetilde{G} \neq G$.

Since automorphisms preserve neighborhoods and vertices of G are identified in $\widetilde{G}$ exactly when they have identical neighborhoods, every automorphism α of G induces an automorphism $\widetilde{\alpha}$ of $\widetilde{G}$ given by $\widetilde{\alpha}([x]) = [\alpha(x)]$. However, it need not be the case that all automorphisms of $\widetilde{G}$ arise in this way. The only nontrivial automorphism of G in Figure 3 is the one interchanging the twin vertices x and y, which induces the identity on $\widetilde{G} = P_4$. However, P_4 has a nontrivial automorphism.

Throughout the rest of this section, we will use $\widetilde{G}$ to denote the quotient graph of a graph G and $\widetilde{\alpha}$ to denote an automorphism of a quotient graph. All sets of vertices with tilde notation will represent sets of vertices in a quotient graph.

We call a minimum size subset of $V(G)$ containing at least one vertex from every pair of twin vertices a *minimum twin cover*. In other words, a minimum twin cover contains precisely all but one vertex from every equivalence class. Recall that every determining set must also contain all but one vertex in each collection of mutual twins. That is, every determining set must contain a minimum twin cover. Thus, if T is a minimum twin cover of G then $|T| \leq \mathrm{Det}(G)$. Denote the image of T under the quotient map as $\widetilde{T}$. For example, for the graph G in Figure 3, if $T = \{y\}$, then $\widetilde{T} = \{[x]\} = \{[y]\}$. Note $\widetilde{T}$ is precisely the set of non-singleton equivalence classes.

Lemma 4.1. *Let T be a minimum twin cover of G. Suppose $\widetilde{\alpha}$ is an automorphism of $\widetilde{G}$ that fixes $\widetilde{T}$. Then there exists an automorphism α of G that fixes T and that induces $\widetilde{\alpha}$.*

Proof. Since $\widetilde{T}$ is precisely the set of non-singleton equivalence classes, and $\widetilde{\alpha}$ fixes $\widetilde{T}$, $\widetilde{\alpha}$ can only map singleton equivalence classes to other singleton equivalence classes. Thus, we can define

$$\alpha(x) = \begin{cases} x, & \text{if } [x] \text{ is not a singleton,} \\ y, & \text{if } [x] \text{ is a singleton and } \widetilde{\alpha}([x]) = [y]. \end{cases}$$

It is straightforward to check that α is an automorphism of G that fixes T, and that $\widetilde{\alpha}([v]) = [\alpha(v)]$ for all $v \in V(G)$. Thus, $\widetilde{\alpha}$ is induced by α. $\square$

Note that if $\widetilde{\alpha}$ is an automorphism of $\widetilde{G}$ that does not fix $\widetilde{T}$, then the automorphism may not extend to G. This is the case for the automorphism $\widetilde{\alpha}$ of $\widetilde{G}$ given by the vertical symmetry of the drawing in Figure 3. In the case that a minimum twin cover is a determining set, we get the following corollary.

Corollary 4.2. *Let T be a minimum twin cover of G. If T is a determining set for G, then $\widetilde{T}$ is a determining set for $\widetilde{G}$.*

Proof. Let $\widetilde{\alpha}$ be an automorphism of $\widetilde{G}$ that fixes each element of $\widetilde{T}$. By Lemma 4.1, this gives an automorphism α of G that fixes T. Since T is a determining set for G, we have that α is the identity. Since $\widetilde{\alpha}$ is induced by α, we get that $\widetilde{\alpha}$ is the identity. $\square$

Note that any superset of a determining set is still determining. Thus, if $\widetilde{D}$ is a determining set for $\widetilde{G}$, then $\widetilde{T} \cup \widetilde{D}$ is a determining set for $\widetilde{G}$ containing $\widetilde{T}$.

Theorem 4.3. *Let T be a minimum twin cover of G. Among all determining sets for $\widetilde{G}$ containing $\widetilde{T}$, let $\widetilde{S}$ be one of minimum size. Let*

$$S = T \cup \{x \in V(G) \mid [x] \in \widetilde{S} \setminus \widetilde{T}\}.$$

Then S is a minimum size determining set for G. In particular, if $\widetilde{S} = \widetilde{T}$, then T is a minimum size determining set for G.

Proof. Let α be an automorphism of G that fixes each vertex in S. We will show that α must be the trivial automorphism. Let $\widetilde{\alpha}$ be the automorphism of $\widetilde{G}$ induced by α. Observe that since α fixes each vertex in S, $\widetilde{\alpha}$ fixes each vertex in $\widetilde{S}$. Since $\widetilde{S}$ is a determining set for $\widetilde{G}$, $\widetilde{\alpha}$ must be the identity on $\widetilde{G}$. This implies that, in particular, $\widetilde{\alpha}$ fixes all singleton equivalence classes and so α fixes all vertices that do not have a twin. Since $\widetilde{\alpha}$ also fixes non-singleton equivalence classes, α preserves equivalence classes of twins. So, if a vertex x has a twin, then either $x \in T \subseteq S$ and is fixed by α, or $x \notin T$, but all of the twins of x are. In the latter case, since x can only be mapped to one of its twins, x is fixed as well. Thus α is the identity and S is a determining set for G.

We will now show S is a minimum size determining set for G. Suppose that R is a minimum size determining set for G such that $|R| < |S|$. We will show $|\widetilde{R}| < |\widetilde{S}|$ and $\widetilde{R} = \{[r] \mid r \in R\}$ is a determining set for $\widetilde{G}$ containing $\widetilde{T}$, a contradiction. Since R must contain at least one vertex of every twin pair, we may assume without loss of generality that $T \subseteq R$, by swapping a vertex with one of its twins if necessary. By definition $T \subseteq S$ as well, so $|R \backslash T| < |S \backslash T|$.

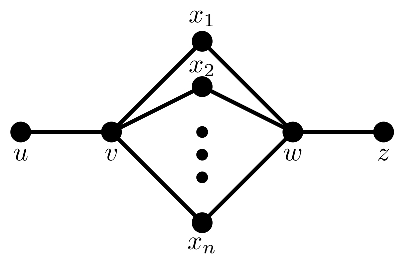

Figure 4: A graph G for which no minimum twin cover is determining.

Because R is of minimum size, no vertex in $R \backslash T$ has a twin in G, and therefore the vertices in $\widetilde{R} \backslash \widetilde{T}$ are singleton equivalence classes. Thus, $|\widetilde{R} \backslash \widetilde{T}| = |R \backslash T|$. The same is true for $S \backslash T$ by definition of S, and so $|S \backslash T| = |\widetilde{S} \backslash \widetilde{T}|$. All together we have that $|\widetilde{R} \backslash \widetilde{T}| < |\widetilde{S} \backslash \widetilde{T}|$. Since $\widetilde{R}$ and $\widetilde{S}$ each contain $\widetilde{T}$, we now conclude that $|\widetilde{R}| < |\widetilde{S}|$.

To show $\widetilde{R}$ is a determining set for $\widetilde{G}$, let $\widetilde{\alpha}$ be an automorphism of $\widetilde{G}$ that fixes each element of $\widetilde{R}$, and hence, each element of $\widetilde{T}$. Let α be the automorphism corresponding to $\widetilde{\alpha}$ given in the proof of Lemma 4.1. Since $\widetilde{\alpha}$ fixes $\widetilde{R}$ it fixes both $\widetilde{T}$ and the singleton equivalence classes in $\widetilde{R} \backslash \widetilde{T}$. Because $\widetilde{\alpha}$ fixes the singleton equivalence classes in $\widetilde{R}$, α fixes the vertices in R that do not have twins. By Lemma 4.1, α also fixes the vertices of T. Thus α fixes the vertices in R. Since R is determining, α is the identity on G. So $\widetilde{\alpha}$ is the identity on $\widetilde{G}$, meaning $\widetilde{R}$ is a determining set and we have reached a contradiction.

Finally, if $\widetilde{S} = \widetilde{T}$, then $\widetilde{S} \setminus \widetilde{T}$ is empty, so $S = T$ and we conclude that T is a minimum size determining set for G. □

From Theorem 4.3, when the image $\widetilde{T}$ of minimum twin cover T of G yields a determining set for $\widetilde{G}$, then $\operatorname{Det} G = |T|$.

Example 4.4. For the graph G in Figure 3, $T = \{y\}$ is a minimum twin cover. Since $\widetilde{T} = \{[x]\}$ is a minimum size determining set for $\widetilde{G}$, we have $\widetilde{S} = \widetilde{T}$, and $S = T = \{y\}$ is a minimum size determining set for G. In addition, increasing the size of $[x]$ by adding additional twins of x gives an infinite family for which $\operatorname{Det}(G) = |T|$.

Example 4.5. Let G be the graph in Figure 4, with $T = \{x_2, \dots x_n\}$. Then $\widetilde{G} = P_5$ and $\widetilde{T}$ consists only of its central vertex. This is not a determining set, although any other singleton subset of $V(\widetilde{G})$ is. In this case, $\widetilde{S}$ is any two-vertex set containing the central vertex; for example, if $\widetilde{S} = \{[x_1], [v]\}$. Then Theorem 4.3 states that $S = \{x_2, \dots x_n, v\}$ is a minimum size determining set for G. This family of examples has $\operatorname{Det}(G) = |T| + \operatorname{Det}(\widetilde{G})$.

Theorem 4.3 yields natural bounds on $\operatorname{Det}(G)$ in terms of $|T|$ and $\operatorname{Det}(\widetilde{G})$.

Corollary 4.6. *Let T be a minimum twin cover of G. Then*

$$|T| \leq \mathrm{Det}(G) \leq |T| + \mathrm{Det}(\widetilde{G}),$$

with both bounds sharp.

Proof. Let $\widetilde{R}$ be a minimum size determining set for $\widetilde{G}$. Then $\mathrm{Det}(\widetilde{G}) = |\widetilde{R}|$ and $\widetilde{T} \cup \widetilde{R}$ is a determining set for $\widetilde{G}$ containing $\widetilde{T}$. By the first part of the proof of Theorem 4.3, $R = T \cup \{x \in V(G) \mid [x] \in \widetilde{R} \backslash \widetilde{T}\}$ is a determining set for G. Moreover, R has size at least $|T|$ and at most $|T| + |\widetilde{R}| = |T| + \mathrm{Det}(\widetilde{G})$.

Example 4.4 gives sharpness in the lower bound while Example 4.5 gives sharpness in the upper bound. □

The preceding corollary can be used to establish bounds on the determining number of $\mu^{(t)}(G)$ in the case where G has twins. To do so, we must investigate how applying the generalized Mycielski construction affects the size of a minimum twin cover as well as the relationship between $\mathrm{Det}(\widetilde{G})$ and $\mathrm{Det}(\widetilde{\mu^{(t)}(G)})$.

Lemma 4.7. *Let G have no isolated vertices and let T be a minimum twin cover of G. Then the set consisting of vertices in T and all of their shadows,*

$$T^{(t)} = \{u_i^s \mid v_i \in T, \, 0 \leq s \leq t\},$$

is a minimum twin cover of $\mu^{(t)}(G)$ of size $(t+1)|T|$.

Proof. By Observations 2.7, 2.8, and 2.9, twin vertices in $\mu^{(t)}(G)$ must be shadows at level s of twins in G for some $0 \leq s \leq t$. Thus, if T contains all but one vertex from any set of mutual twin vertices in G, then the copy of T at level s contains all but one vertex from each set of mutual twins in $\mu^{(t)}(G)$ at level s. So $T^{(t)}$ is a minimum twin cover. □

The following lemma proves that the processes of applying the generalized Mycielski construction commutes with the process of taking the quotient graph. Figure 5 gives an example of this for the graphs $K_{1,3}$ and $\mu^{(t)}(K_{1,3})$ with $t = 1, 2$, and for their quotient graphs.

Lemma 4.8. *If G has no isolated vertices, then $\mu^{(t)}(\widetilde{G}) = \widetilde{\mu^{(t)}(G)}$.*

Proof. By Observations 2.7, 2.8, and 2.9, two vertices are twin in $\mu^{(t)}(G)$ if and only if either they are twin vertices in G or they are shadows at level s of twin vertices in G. In terms of our equivalence relation, $v_i \sim v_j$ in G if and only if $u_i^s \sim u_j^s$ for all $0 \leq s \leq t$ in $\mu^{(t)}(G)$. This allows us to map the shadows at level s of $[v_i]$ in $\mu^{(t)}(\widetilde{G})$ to $[u_i^s]$ in $\widetilde{\mu^{(t)}(G)}$. Additionally, if w is the shadow master in $\mu^{(t)}(G)$, then we map the shadow master of $\mu^{(t)}(\widetilde{G})$ to $[w]$ in $\widetilde{\mu^{(t)}(G)}$. It it straightforward to verify that this map preserves both adjacencies and non-adjacencies. Therefore, $\mu^{(t)}(\widetilde{G}) = \widetilde{\mu^{(t)}(G)}$. □

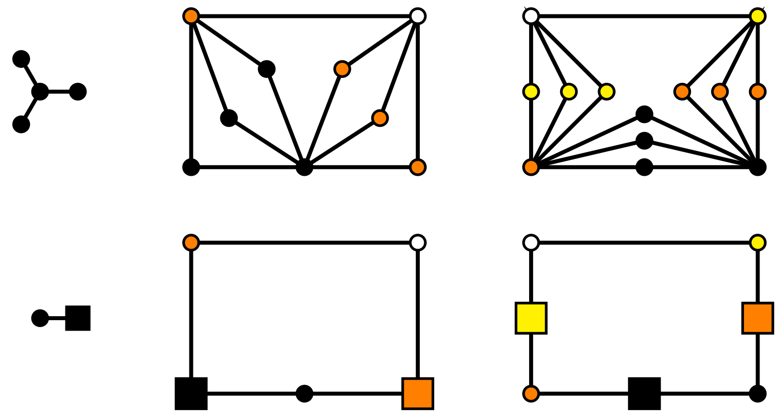

Figure 5: The graphs $K_{1,3}$ and $\mu^{(t)}(K_{1,3})$ for $t = 1, 2$, and their quotient graphs, with the collapsed twin vertices shown as squares.

If $G \neq K_{\ell,m}$, then we can extend Lemma 4.8 to say something about $\mathrm{Det}(\widetilde{\mu^{(t)}(G)})$. We only address the case $G \neq K_{\ell,m}$ because it is the only case to which we refer in subsequent proofs.

Lemma 4.9. *If $G \neq K_{\ell,m}$ and has no isolated vertices, then*

$$\mathrm{Det}(\widetilde{G}) = \mathrm{Det}(\widetilde{\mu^{(t)}(G)}).$$

Proof. Note that by Lemma 4.8, $\mathrm{Det}(\widetilde{\mu^{(t)}(G)}) = \mathrm{Det}(\mu^{(t)}(\widetilde{G}))$. The assumption that $G \neq K_{\ell,m}$ implies that $\widetilde{G} \neq K_2$. Moreover, $\widetilde{G}$ is twin-free with no isolated vertices and so by Theorem 3.1(ii), $\mathrm{Det}(\mu^{(t)}(\widetilde{G})) = \mathrm{Det}(\widetilde{G})$. $\square$

If G has a minimum twin cover that is a determining set, then we can state the determining number of $\mu^{(t)}(G)$ in terms of the determining number of G. This generalizes Theorem 3.1(ii).

Theorem 4.10. *Let G be a graph with no isolated vertices. If $G \neq \widetilde{G}$ and G has a minimum twin cover that is a determining set, then*

$$\mathrm{Det}(\mu^{(t)}(G)) = (t+1)\,\mathrm{Det}(G).$$

Proof. First we consider the case where $G = K_{\ell,m}$ for $1 \leq \ell \leq m$. Then $\widetilde{G} = K_2$ and $G \neq \widetilde{G}$ together imply that $m \geq 2$. Let $\{v_1, \ldots, v_\ell\}$ and $\{v_{\ell+1}, \ldots v_{\ell+m}\}$ be the partite sets in $V(K_{\ell,m})$. Since every minimum twin cover contains all but one vertex from each partite set, without loss of generality, we can choose $T = V(K_{\ell,m}) \setminus \{v_\ell, v_{\ell+m}\}$ as a minimum twin cover of G. It is straightforward to verify that T is a minimum size determining set for $K_{\ell,m}$ and, therefore, $|T| = \mathrm{Det}(G)$.

By Lemma 4.7, $T^{(t)}$ is a minimum twin cover of $\mu^{(t)}(G)$ of size $(t+1)|T|$. We will show that $T^{(t)}$ is a minimum size determining set for $\mu^{(t)}(G)$. Let $\widehat{\alpha}$ be an automorphism of $\mu^{(t)}(G)$ that fixes the vertices in $T^{(t)}$.

Recall that $\ell \geq 1$ and $m \geq 2$. Therefore, all vertices u_i^j for $\ell+1 \leq i \leq \ell+m$ and $0 \leq j \leq t$ are in non-singleton equivalence classes. Since twin relationships are preserved by automorphisms, if a vertex x is fixed by an automorphism, then its set of twins, $[x]$, is preserved setwise. Further, if an automorphism fixes all but one of the vertices in a non-singleton equivalence class $[x]$, it must then fix the vertices of $[x]$ pointwise. This implies that $\widehat{\alpha}$ fixes all vertices of $\mu^{(t)}(G)$ except possibly for those in a singleton class, which are w, and, if $\ell = 1$, the vertices $u_\ell^0, \ldots, u_\ell^t$. Since for $0 \leq i \leq t-1$, u_ℓ^i is only adjacent to shadow vertices of the other partite set, each of $u_\ell^0, \ldots, u_\ell^{t-1}$ has a unique neighborhood in $\mu(G)$ that is entirely fixed by $\widehat{\alpha}$, and therefore must also be fixed by $\widehat{\alpha}$. The vertex u_ℓ^t is adjacent to vertices $u_{\ell+1}^{t-1}, \ldots, u_{\ell+m}^{t-1}$, all of which are fixed by $\widehat{\alpha}$. The vertex w is adjacent to none of these. Therefore, $\widehat{\alpha}$ must also fix u_ℓ^t and w, and thus it fixes all vertices of $\mu^{(t)}(G)$. Thus the minimum twin cover $T^{(t)}$ is determining, and so is a minimum size determining set.

Now suppose $G \neq K_{\ell,m}$. We will apply Theorem 4.3 to $\mu^{(t)}(G)$. Let $G \neq K_{\ell,m}$ be a graph with no isolated vertices. By Lemma 4.7, $T^{(t)}$ is a minimum twin cover of $\mu^{(t)}(G)$. We seek a determining set for the quotient graph $\widetilde{\mu^{(t)}(G)}$ containing $\widetilde{T^{(t)}}$ that is of minimum size. By Lemma 4.8, $\widetilde{\mu^{(t)}(G)} = \mu^{(t)}(\widetilde{G})$. Moreover, by Observations 2.7, 2.8, and 2.9, $\widetilde{T^{(t)}} = \widetilde{T}^{(t)}$.

By Corollary 4.2, since T is a determining set for G, we know $\widetilde{T}$ is a determining set for $\widetilde{G}$. Since $\widetilde{G} \neq K_2$ is twin-free and has no isolated vertices, by Theorem 3.1(ii), any determining set for $\widetilde{G}$ is also a determining set for $\mu^{(t)}(\widetilde{G})$. Thus $\widetilde{T}$ is a determining set for $\mu^{(t)}(\widetilde{G})$. Since $\widetilde{T}^{(t)}$ contains $\widetilde{T}$, it is also a determining set for $\mu^{(t)}(\widetilde{G})$. So $\widetilde{T}^{(t)}$ is a determining set for $\widetilde{\mu^{(t)}(G)} = \mu^{(t)}(\widetilde{G})$ containing $\widetilde{T^{(t)}} = \widetilde{T}^{(t)}$ of minimum size. Therefore, by Theorem 4.3, $T^{(t)}$ is a minimum size determining set for $\mu^{(t)}(G)$. Thus $\mathrm{Det}(\mu^{(t)}(G)) = (t+1)\,\mathrm{Det}(G)$. □

If G does not have a minimum twin cover that is also a determining set, then we cannot give an exact relationship between $\mathrm{Det}(G)$ and $\mathrm{Det}(\mu^{(t)}(G))$. However, for all graphs with twins, we can combine our results to obtain upper and lower bounds on the determining number of the generalized Mycielskian of G in terms of the size of a minimum twin cover of G and the determining number of the quotient graph $\widetilde{G}$. Note that the result shows that if G has twins, then as t gets large, the determining number of $\mu^{(t)}(G)$ becomes dominated by the size of a minimum twin cover for $\mu^{(t)}(G)$.

Theorem 4.11. *Let T be a minimum twin cover of a graph G with no isolated vertices and $G \neq \widetilde{G}$. Then*

$$(t+1)|T| \leq \mathrm{Det}(\mu^{(t)}(G)) \leq (t+1)|T| + \mathrm{Det}(\widetilde{G}),$$

with both bounds sharp.

Proof. If $G = K_{\ell,m}$, then $G \neq \widetilde{G}$ tells us that $\ell \geq 1$ and $m \geq 2$. In this case, $|T| = \mathrm{Det}(G)$ and so by Theorem 4.10 we have equality in the lower bound.

Suppose $G \neq K_{\ell,m}$. By Lemma 4.7, $T^{(t)}$ is a minimum twin cover of $\mu^{(t)}(G)$ of size $(t+1)|T|$. By Corollary 4.6,

$$(t+1)|T| \leq \mathrm{Det}(\mu^{(t)}(G)) \leq (t+1)|T| + \mathrm{Det}(\widetilde{\mu^{(t)}(G)}).$$

Since $G \neq K_{\ell,m}$, by Lemma 4.9, $\mathrm{Det}(\widetilde{G}) = \mathrm{Det}(\widetilde{\mu^{(t)}(G)})$. The upper bound is achieved in the case where G is the graph in Figure 4. □

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