

Geometric Graph Homomorphisms

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Abstract: A *geometric graph* \overline{G} is a simple graph drawn on points in the plane, in general position, with straightline edges. A *geometric homomorphism* from \overline{G} to \overline{H} is a vertex map that preserves adjacencies and crossings. This work proves some basic properties of geometric homomorphisms and defines the *geochromatic number* as the minimum n so that there is a geometric homomorphism from \overline{G} to a geometric n -clique. The geochromatic number is related to both the chromatic number and to the minimum number of plane layers of \overline{G} . By providing an infinite family of bipartite geometric graphs, each of which is constructed of two plane layers, which take on all possible values of geochromatic number, we show that these relationships do not determine the geochromatic number. This article also gives necessary (but not sufficient) and sufficient (but not necessary) conditions for a geometric graph to have geochromatic number at most four. As a corollary, we get precise criteria for a bipartite geometric graph to have geochromatic number at most four. This article also gives criteria for a geometric graph to be homomorphic to certain geometric realizations of $K_{2,2}$ and $K_{3,3}$.

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1. INTRODUCTION

Much work has been done studying graph homomorphisms. A graph homomorphism $f:G\rightarrow H$ is a vertex map that preserves adjacencies (but not necessarily non-adjacencies). If such a map exists, we write $G\rightarrow H$ and say “ G is homomorphic to H ”. As of this writing, there are 166 research articles and books addressing graph homomorphisms. One of the most natural, and widespread, uses for homomorphisms is to generalize graph colorings—proper colorings, circular colorings, fractional colorings, oriented and acyclic colorings. A good reference for graph homomorphisms is [2].

This article introduces homomorphisms of geometric graphs. A geometric graph \overline{G} is a simple graph drawn in the plane, on points in general position, with straightline edges. What we care about in a geometric graph is which pairs of vertices are adjacent and which pairs of edges cross. In particular, two geometric graphs are said to be isomorphic if there is a bijection between their vertex sets that preserves adjacencies, non-adjacencies, crossings, and non-crossings.

A natural way to extend the idea of abstract graph homomorphism to the context of geometric graphs is to define a geometric homomorphism as a vertex map $f:\overline{G}\rightarrow\overline{H}$ that preserves both adjacencies and crossings (but not necessarily non-adjacencies or non-crossings). If such a map exists, we write $\overline{G}\rightarrow\overline{H}$ and say “ \overline{G} is (geometrically) homomorphic to \overline{H} ”. There are many similarities between abstract graph homomorphisms and geometric graph homomorphisms, but there are also great contrasts. Results that are straightforward in abstract graph homomorphism theory can become complex in geometric graph homomorphism theory.

In abstract graph homomorphism theory (of simple graphs), two vertices cannot be identified under any homomorphism if and only if they are adjacent. However, in Section 2 we see that there are various reasons why two vertices might not be able to be identified under any geometric homomorphism: if they are adjacent, if they are involved in a common edge crossing, if they are endpoints of an odd-length path each edge of which is crossed by a common edge, and if they are endpoints of a path of length two whose edges cross all edges of an odd length cycle.

In abstract graph homomorphism theory (of simple graphs), the only type of graph that is not homomorphic to a graph on fewer vertices is a complete graph. In geometric homomorphism theory, there are many graphs other than complete graphs that are not homomorphic to any geometric graph on fewer vertices. Some examples are given in Section 3.

In abstract graph homomorphism theory, every graph on n vertices is homomorphic to K_n . This is not true for geometric graphs. In fact, two different geometric realizations of the same abstract graph are not necessarily homomorphic to each other. For example, consider the two geometric realizations of K_6 given in Figure 1. The first has a vertex with all incident edges crossed, the second does not; this can be used to prove that there is no geometric homomorphism from the first to the second. The second has more crossings than the first; this can be used to prove that there is no geometric homomorphism from the second to the first.

One definition for the *chromatic number* of a graph G , denoted by $\chi(G)$, is the smallest integer n so that $G\rightarrow K_n$. By the transitivity of homomorphisms, if $G\rightarrow H$, then $\chi(G)\leq\chi(H)$. We analogously define the *geochromatic number* of a geometric graph \overline{G} , denoted by $X(\overline{G})$, to be the smallest integer n so that $\overline{G}\rightarrow\overline{K}_n$, for some

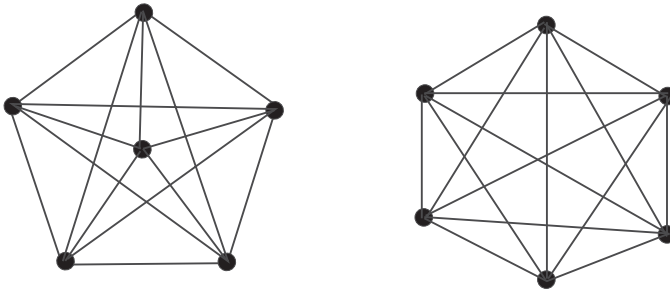


FIGURE 1. Homomorphically distinct realizations of K_6 .

geometric n -clique \overline{K}_n . We immediately get that $\overline{G} \rightarrow \overline{H}$ implies $X(\overline{G}) \leq X(\overline{H})$. However, there are other parameters whose bounds are preserved by geometric homomorphisms. Define the *thickness* of a geometric graph, denoted by $\theta(\overline{G})$, to be the minimum number of plane layers of \overline{G} . Note that the thickness of a geometric graph \overline{G} is not quite the same as the thickness of the abstract graph G (which is the minimum over all geometric realizations). We see in Section 2 that $\overline{G} \rightarrow \overline{H}$ implies both $\chi(G) \leq \chi(H)$ and $\theta(\overline{G}) \leq \theta(\overline{H})$. This may lead one to hope that knowledge of $\chi(G)$ and $\theta(\overline{G})$ would give, or at least bound, the geochromatic number of \overline{G} . That this is not the case is shown in Section 3 by finding an infinite family of bipartite, thickness-2 geometric graphs whose geochromatic numbers are not bounded. Section 4 delves further into bipartite, thickness-2 geometric graphs, giving criteria for a geometric graph to be homomorphic to specific geometric realizations of $K_{2,2}$ and $K_{3,3}$. Section 5 provides a set of necessary conditions and a set of sufficient conditions for a geometric graph to have geochromatic number at most four. It also gives examples that show that neither set of conditions is both necessary and sufficient. As a corollary though, we do get precise criteria for a bipartite geometric graph to have geochromatic number at most four.

2. BASICS

Let \overline{G} be a geometric graph with underlying (simple) abstract graph G . In particular, $V(\overline{G})$ is a set of points in general position in \mathbb{R}^2 and an edge $uv \in E(\overline{G})$ is the line segment joining the points u and v . Two edges uv and xy are said to *cross* if the interiors of the line segments from u to v and from x to y have non-empty intersection. This occurs precisely when the vertices u, x, v , and y are (in cyclic order) the vertices of a convex quadrilateral. See [3, 4] for background on geometric graphs. We call a geometric graph with no crossings a *plane* geometric graph.

Recall that a *graph isomorphism* $\phi: G \rightarrow H$ is a vertex bijection $\phi: V(G) \rightarrow V(H)$ such that u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H . A relaxation of this definition gives us a *graph homomorphism* $f: G \rightarrow H$ as a vertex map $f: V(G) \rightarrow V(H)$ such that if u and v are adjacent in G , then $f(u)$ and $f(v)$ are adjacent in H . These concepts can be extended to geometric graphs.

Definition 1. Let \overline{G} and \overline{H} be geometric graphs.

1. A *geometric isomorphism* $\phi: \overline{G} \rightarrow \overline{H}$ is an isomorphism $\phi: G \rightarrow H$ such that edges uv and xy cross in \overline{G} if and only if edges $\phi(u)\phi(v)$ and $\phi(x)\phi(y)$ cross in \overline{H} .

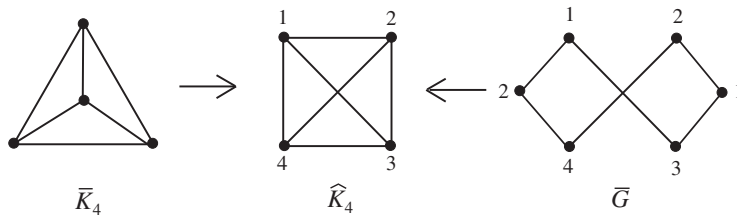


FIGURE 2. Two geometric homomorphisms.

2. A *geometric homomorphism* $f: \overline{G} \rightarrow \overline{H}$ is a homomorphism $f: G \rightarrow H$ such that if edges uv and xy cross in \overline{G} , then edges $f(u)f(v)$ and $f(x)f(y)$ cross in \overline{H} .

Example 1. The first two graphs in Figure 2 are different geometric realizations of K_4 . Up to geometric isomorphism, these are the only realizations of K_4 . Denote by \widehat{K}_4 the thickness-2 realization and by \overline{K}_4 the plane realization. Note that any vertex bijection from \overline{K}_4 to \widehat{K}_4 is a geometric homomorphism but that there is no geometric homomorphism from \widehat{K}_4 to \overline{K}_4 . Note also that the vertex map implied by the labelings of \widehat{K}_4 and \overline{G} in Figure 2 is a geometric homomorphism $\overline{G} \rightarrow \widehat{K}_4$.

Much of the notation and terminology used here mirrors that used for abstract graph homomorphisms as found in [2]. We say that \overline{G} is *geometrically homomorphic* to \overline{H} if there exists a geometric homomorphism from \overline{G} to \overline{H} , and denote this by $\overline{G} \rightarrow \overline{H}$. To ease terminology, we often drop the term *geometric*. That is, a homomorphism between geometric graphs is, by definition, a geometric homomorphism.

It is immediate from the definition that $\overline{G} \rightarrow \overline{H}$ implies $G \rightarrow H$. Furthermore, if \overline{G} is a plane geometric graph, then any abstract graph homomorphism $f: G \rightarrow H$ is also a geometric homomorphism $f: \overline{G} \rightarrow \overline{H}$. It is easy to see that the property of being geometrically homomorphic is transitive. More formally, if $f: \overline{G} \rightarrow \overline{H}$ and $g: \overline{H} \rightarrow \overline{K}$ are geometric homomorphisms, then so is $g \circ f: \overline{G} \rightarrow \overline{K}$.

The following observations are used throughout this article.

Observation 1. Adjacent vertices cannot be identified by any geometric homomorphism.

Observation 2. Endpoints of edges that cross cannot be identified by any geometric homomorphism.

Proof. Since crossing edges are preserved by geometric homomorphism and in a geometric graph no pair of edges with a common endpoint can cross, no geometric homomorphism can identify endpoints of edges that cross. ■

Observation 3. The endpoints of an odd-length path cannot be identified by any geometric homomorphism if there is a single edge crossing all the edges of the path.

Proof. Let \overline{G} be a geometric graph containing an odd-length path $\overline{P} = v_1 v_2 \dots v_{2r}$, each of whose edges is crossed by edge xy . Further, let $f: \overline{G} \rightarrow \overline{H}$ be a geometric homomorphism. Since f is a geometric homomorphism, for each $i = 1, \dots, 2r - 1$, the edge $f(v_i)f(v_{i+1})$ crosses $f(x)f(y)$. Thus, $f(v_i)$ and $f(v_{i+1})$ must be on opposite sides of the line of \mathbb{R}^2 that contains the edge $f(x)f(y)$. Thus $f(v_1), f(v_3), \dots, f(v_{2r-1})$ are on one

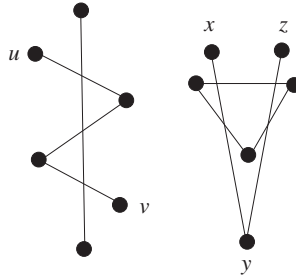


FIGURE 3. Vertices that cannot be identified.

side of this line and $f(v_2), f(v_4), \dots, f(v_{2r})$ are on the other side. In particular, $f(v_1)$ and $f(v_{2r})$ are on opposite sides of the line and therefore cannot be equal. ■

Thus in Figure 3, the vertices u and v cannot be identified by any geometric homomorphism.

Observation 4. The endpoints of a path of length two cannot be identified by any geometric homomorphism if its edges cross all edges of an odd-length cycle.

Proof. Let \bar{G} be a geometric graph containing path $\bar{P}=xyz$ and cycle $\bar{C}_{2r+1}=v_1 \dots v_{2r+1}$. Assume that together the edges xy and yz cross all edges of C_{2r+1} . Let $f: \bar{G} \rightarrow \bar{H}$ be a geometric homomorphism and suppose that $f(x)=f(z)$. Then the image of \bar{P} in \bar{H} is a single edge. Recall that the image of an odd-length cycle under (abstract) homomorphism contains an odd-length cycle. Since f preserves crossings, edge $f(x)f(y)=f(z)f(y)$ crosses each edge of the image of \bar{C}_{2r+1} . Thus, as in the proof of Observation 3, $f(v_1), f(v_3), \dots, f(v_{2r+1})$ are on one side of line determined by $f(x)f(y)=f(y)f(z)$, and $f(v_2), f(v_4), \dots, f(v_{2r})$ are on the other. However, since there is an edge from v_1 to v_{2r+1} in \bar{G} which crosses either xy or yz (or both), the edge $f(v_1)f(v_{2r+1})$ must cross $f(x)f(y)$ in \bar{H} . Therefore, $f(v_1)$ must be on the opposite side of the line $f(x)f(y)$ from $f(v_{2r+1})$. This contradicts our previous conclusion. Thus $f(x) \neq f(z)$. ■

Thus in Figure 3, the vertices x and z cannot be identified by any geometric homomorphism.

3. THE GEOCHROMATIC NUMBER

Recall that a *proper vertex coloring* of a graph G (or geometric graph) is a labeling of the vertices so that adjacent vertices have different labels and that the *chromatic number* $\chi(G)$ is the minimum number of color required for a proper vertex coloring. Also recall that if $f: G \rightarrow H$ is an (abstract) graph homomorphism where H is a simple graph, then for any $v \in V(H)$, the preimage $f^{-1}(v)$ must be an independent set of vertices in G . More generally, the preimage of any independent vertex set in H is independent in G . Since a proper vertex coloring of a graph is a partition of the vertices into independent sets, we can “pull back” any proper vertex coloring of H to a proper vertex coloring of G . This gives us the well-known result that $G \rightarrow H$ implies $\chi(G) \leq \chi(H)$. There is a similar result for the following coloring invariant.

Definition 2. An edge coloring of a geometric graph \overline{G} is called a *thickness edge coloring* if no two edges of the same color cross. The *thickness* of \overline{G} , denoted by $\theta(\overline{G})$, is the minimum number of colors required for a thickness edge coloring of \overline{G} .

Recall that the geometric thickness of an abstract graph G is the minimum $\theta(\overline{G})$ over all geometric realizations \overline{G} of G . We do not discuss geometric thickness of abstract graphs in this article, but one should keep in mind that though these are related invariants, they are not identical.

Proposition 1. If $\overline{G} \rightarrow \overline{H}$, then $\theta(\overline{G}) \leq \theta(\overline{H})$.

Proof. Let $f: \overline{G} \rightarrow \overline{H}$ be a geometric homomorphism. Note that for any edge $e \in E(\overline{H})$, the preimage $f^{-1}(e)$ must be a non-crossing set of edges (plane subgraph) of \overline{G} . More generally, the preimage of any plane subgraph of \overline{H} is a plane subgraph of \overline{G} . Since a thickness edge coloring is a partition of the edges of a geometric graph into plane subgraphs, we can ‘pull back’ a thickness edge coloring of \overline{H} to a thickness edge coloring of \overline{G} . ■

Thus we have that if \overline{G} is homomorphic to \overline{H} , then $\chi(G) \leq \chi(H)$ and $\theta(\overline{G}) \leq \theta(\overline{H})$.

Another natural way to phrase the definition of the chromatic number of a graph G , $\chi(G)$, is as the smallest positive integer n such that $G \rightarrow K_n$. This extends naturally to the following.

Definition 3. Let \overline{G} be a geometric graph. We say that \overline{G} is *n-geocolorable* if $\overline{G} \rightarrow \overline{K}_n$, for some geometric realization of K_n . The *geochromatic number* of \overline{G} , denoted by $X(\overline{G})$, is the smallest positive integer n such that \overline{G} is *n-geocolorable*.

As a mnemonic device, X may be thought of as a straightline χ .

Note that by Proposition 1, if $X(\overline{G}) = n$ then for some \overline{K}_n , $\overline{G} \rightarrow \overline{K}_n \implies \theta(\overline{G}_n) \leq \theta(\overline{K}_n)$. That is, the geochromatic number of \overline{G} must be large enough not only to accommodate the adjacency relationships among vertices of \overline{G} , but also to accommodate the crossing relationships among its edges.

Note that the homomorphic definition of chromatic number of a graph is equivalent to “the smallest n so that the vertices of G can be colored with n colors so that distinct colors are given to all vertex pairs corresponding to edges”. Similarly, it is natural to hope that the definition of $X(\overline{G})$ is equivalent to “the smallest integer n so that the vertices of \overline{G} can be colored with n colors so that distinct colors are given to all vertex pairs corresponding to edges and to all vertex quadruples corresponding to crossing edges”. However, this is not the case. Denote by \overline{G} the righthand graph in Figure 3. By coloring x and z with the same color in \overline{G} and all other vertices distinct colors, we obtain a five coloring that meets the latter conditions. However, as shown in Observation 4, x and z cannot be identified under any geometric homomorphism. Further, by Observation 2, no other vertex pair of \overline{G} can be identified by any homomorphism since each pair is involved in a common crossing. We conclude that $X(\overline{G}) = 6$. Thus, the latter conditions are weaker than the definition of geochromatic number. In as-yet-unpublished work, Dean and Margea have shown that the difference in the number of colors necessary for these two types of colorings can be arbitrarily large [1].

We can classify geometric graphs with low geochromatic number quite easily. Trivially, $X(\overline{G}) = 1$ if and only if \overline{G} is a null graph. Since the only geometric realization

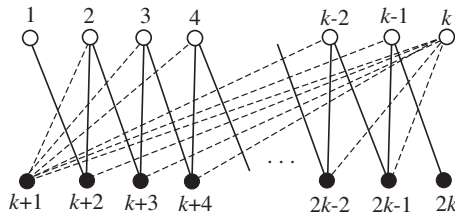


FIGURE 4. A bipartite, thickness-2 geometric graph with $X(\overline{G})=2k$.

of K_2 is plane, if $X(\overline{G})=2$ then \overline{G} is bipartite and thickness-1 by Proposition 1. The converse is obvious. Thus, $X(\overline{G})=2$ if and only if \overline{G} is a bipartite plane geometric graph. Similarly, since the only geometric realization of K_3 is plane, $X(\overline{G})=3$ if and only if \overline{G} is a 3-chromatic plane geometric graph. Since deciding whether $\chi(G) \leq 3$ is an NP-complete problem [5], deciding whether $X(\overline{G}) \leq 3$ is NP-complete also.

Recall from the previous section that there are two geometric realizations of K_4 , one is plane, denoted by \overline{K}_4 , the other is thickness-2 denoted by \widehat{K}_4 . Note that $\overline{G} \rightarrow \overline{K}_4$ if and only if \overline{G} is a plane geometric graph. Since the plane realization is homomorphic to the thickness-2 one, it is sufficient for us to study criteria for $\overline{G} \rightarrow \widehat{K}_4$. By Proposition 1, $X(\overline{G}) \leq 4$ implies $\chi(G) \leq 4$ and $\theta(\overline{G}) \leq 2$. However, the converse is false. That is, the geochromatic number of a 4-colorable, thickness-2 geometric graph is not bounded above by four. Even the geochromatic number of a bipartite, thickness-2 geometric graph is not bounded by four. As the theorem below shows, the geochromatic number of such a graph can be arbitrarily large.

Theorem 1. *The geochromatic number of a bipartite, thickness-2 geometric graph is not bounded.*

Proof. In the following, we construct \overline{G} on $2k$ vertices as in Figure 4. Place k white vertices in a “line”. (Recall that a geometric graph cannot have three vertices on the same line. However, after we finish this construction, we could perturb the vertices slightly so they are in general position and thereby fulfill the definition for a geometric graph. This does not affect any of our arguments.) Label these vertices $1, \dots, k$ in order. Place k black vertices below these, also in a line. Label these vertices $k+1, \dots, 2k$ in order.

For each $i=2, \dots, k-1$, connect i to $k+i$ and $k+i+1$ with a solid edge. Also connect 1 to $k+2$ with a solid edge. Note that none of the solid edges cross each other. Now connect $k+1$ to each of $2, \dots, k$ with a dashed edge and k to each of $k+2, \dots, 2k-1$ with a dashed edge. Note that none of the dashed edges cross each other. Thus by construction, \overline{G} is bipartite and thickness-2.

Let i and j be a pair of arbitrary white vertices and assume that $i < j$. Note that the dashed edge from j to $k+1$ crosses the solid edge from i to $k+i+1$. Since i and j are involved in a common crossing, by Observation 2 they cannot be identified under any geometric homomorphism. A similar argument shows that no pair of black vertices can be identified under any geometric homomorphism.

By Observation 1, since k and $k+1$ are adjacent, they cannot be identified by any geometric homomorphism. Note that the dashed edge $(k+1)k$ crosses each solid edge. Further, every vertex other than $k+1$ and k is incident to a solid edge. Let i

be an arbitrary white vertex other than k . By construction, edge $i(i+k)$ crosses edge $(k+1)k$. Thus, $k+1$ cannot be identified with any white vertex under a geometric homomorphism. The same is true for the vertex k and any black vertex.

Consider a white vertex i and a black vertex j where $i \neq k$ and $j \neq k+1$. By construction, there is an odd-length path \bar{P} of solid edges between i and j , each edge of which is crossed by the edge $k(k+1)$. Thus by Observation 3, no geometric homomorphism can identify i and j .

Thus $X(\bar{G}) \geq 2k$. Since this construction works for any positive integer $k \geq 2$, the geochromatic number of bipartite, thickness-2 geometric graphs is not bounded.

Note that by adding all missing edges to \bar{G} to create \bar{K}_{2k} , we can show that $\bar{G} \rightarrow \bar{K}_{2k}$ and therefore $X(\bar{G}) = 2k$. Also, notice that this proof can be modified to work for odd integers by deleting the vertex $2k$. All arguments are the same. ■

Theorem 1 shows that we need to know more than just the chromatic number and thickness of a non-plane geometric graph in order to determine its geochromatic number. Before addressing the question of which geometric graphs have geochromatic number at most 4, we hone our understanding of geometric homomorphisms in the next section.

4. COMPLETE BIPARTITE GRAPHS

We wish to consider the question of when a geometric graph is homomorphic to a realization of the complete bipartite graph $K_{m,n}$. In the simplest case, note that the only geometric realization of $K_{1,n}$ is a plane geometric graph. It is not hard to show $\bar{G} \rightarrow \bar{K}_{1,n}$ if and only if \bar{G} is 2-geocolorable. The next simplest case is $K_{2,2}$. There are two geometric realizations. One is a plane geometric graph; the other is a thickness-2 realization which we denote by $\widehat{K}_{2,2}$. See Figure 5. Since the plane realization is homomorphic to the thickness-2 one, it is sufficient for us to study criteria for $\bar{G} \rightarrow \widehat{K}_{2,2}$.

To obtain necessary and sufficient conditions for $\bar{G} \rightarrow \widehat{K}_{2,2}$, we require some definitions.

Definition 4. Let \bar{G} be a geometric graph.

1. An edge e is called a *crossing edge* if it crosses another edge. Denote the set of all crossing edges of \bar{G} by E_\times .
2. A vertex v of \bar{G} is called a *crossing vertex* if it is incident to a crossing edge. Otherwise, v is called a *non-crossing vertex*. Denote the set of all crossing vertices of \bar{G} by V_\times .

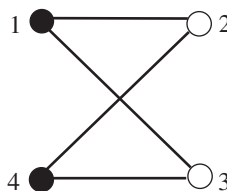


FIGURE 5. $\widehat{K}_{2,2}$.

3. Let the *crossing subgraph*, \overline{G}_\times , be the geometric subgraph of \overline{G} with vertex set V_\times and edge set E_\times .
4. Let $\overline{C}_1, \overline{C}_2, \dots, \overline{C}_m$ denote the connected components of \overline{G}_\times ; these are called the *crossing components* of \overline{G} . (Note that being “connected” here is in the abstract graph theory sense of there being an edge path between every pair of vertices.)
5. Let the *induced crossing subgraph*, $\overline{G}[V_\times]$, be the geometric subgraph induced by the crossing vertices.

Figure 6 illustrates these definitions on a realization of the Grotzch graph.

Lemma 1. *If $\overline{G} \rightarrow \overline{H}$, then $\overline{G}_\times \rightarrow \overline{H}_\times$ and $\overline{G}[V_\times] \rightarrow \overline{H}[V_\times]$.*

Proof. Assume $f: \overline{G} \rightarrow \overline{H}$. Since geometric homomorphisms preserve crossing edges, the images of the crossing vertices of \overline{G} under f is a subset of the crossing vertices of \overline{H} . Hence, we can restrict f appropriately to obtain both $f: \overline{G}_\times \rightarrow \overline{H}_\times$ and $f: \overline{G}[V_\times] \rightarrow \overline{H}[V_\times]$. ■

The example in Figure 7 shows that the converse of Lemma 1 is false. Here, we have two geometric graphs with the same crossing subgraph and induced crossing subgraph. By identifying the two non-crossing vertices, we can see that $\overline{G} \rightarrow \overline{H}$. However, since $3 = \chi(H) > \chi(G) = 2$ we can see that $\overline{H} \not\rightarrow \overline{G}$.

Definition 5. Given a geometric graph \overline{G} , its *crossing component graph*, C_\times , is the graph whose vertices correspond to the crossing components $\overline{C}_1, \overline{C}_2, \dots, \overline{C}_m$ of \overline{G} , with an edge between vertices \overline{C}_i and \overline{C}_j if an edge of \overline{C}_i crosses an edge of \overline{C}_j in \overline{G} .

An example of C_\times is given in Figure 8. Note that C_\times is an abstract graph and is not necessarily simple. If a crossing component is not plane, then C_\times has a loop. However, in this article we only see crossing component subgraphs in situations where

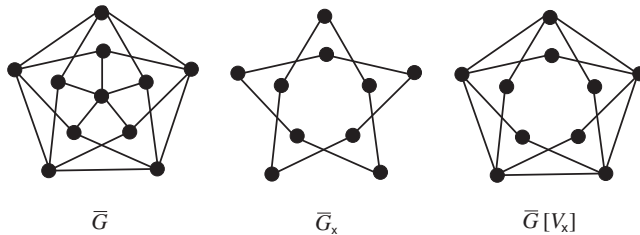


FIGURE 6. Crossing subgraph; induced crossing subgraph.

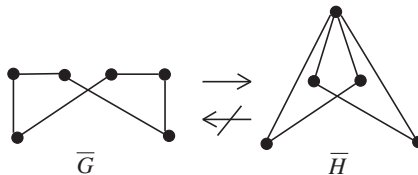


FIGURE 7. $\overline{G}_\times \rightarrow \overline{H}_\times$ does not imply $\overline{G} \rightarrow \overline{H}$.

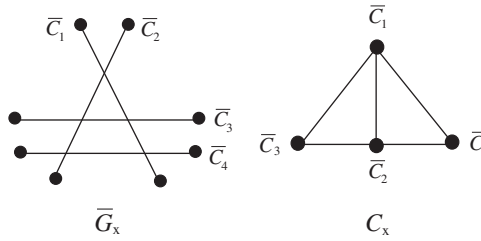


FIGURE 8. Crossing subgraph and crossing component graph.

each crossing component is a plane geometric graph. In these cases, C_\times is a simple graph.

Definition 6. For any subset of vertices $Y = \{\bar{C}_{i_1}, \dots, \bar{C}_{i_r}\}$ of the crossing component graph C_\times , let \bar{G}_Y denote the subgraph of \bar{G} induced by the vertices in $\bar{C}_{i_1} \cup \dots \cup \bar{C}_{i_r}$.

Note that \bar{G}_Y includes any non-crossing edges between vertices in $\bar{C}_{i_1} \cup \dots \cup \bar{C}_{i_r}$. That is, \bar{G}_Y is a subgraph of $\bar{G}[V_\times]$, not necessarily of \bar{G}_\times .

Theorem 2. A geometric graph \bar{G} is homomorphic to $\widehat{K}_{2,2}$ if and only if

1. \bar{G} is bipartite;
2. each crossing component \bar{C}_i of \bar{G} is a plane subgraph;
3. C_\times is bipartite.

Proof. Assume $f: \bar{G} \rightarrow \widehat{K}_{2,2}$ is a geometric homomorphism. Since \bar{G} is homomorphic to $\widehat{K}_{2,2}$, $\chi(\bar{G}) \leq \chi(K_{2,2}) = 2$. Therefore \bar{G} is bipartite.

Label $\widehat{K}_{2,2}$ as in Figure 5. Note that the crossing subgraph of $\widehat{K}_{2,2}$ consists of the disjoint edges 13 and 24. Since geometric homomorphisms preserve connectivity and crossings, each crossing component \bar{C}_i is sent by f to either edge 13 or edge 24. Further, since the preimage of a plane graph under a geometric homomorphism is a plane graph, each crossing component \bar{C}_i is a plane subgraph of \bar{G} .

Partition the vertices of C_\times by setting $U = \{\bar{C}_i \mid f(\bar{C}_i) = 13\}$ and $V = \{\bar{C}_i \mid f(\bar{C}_i) = 24\}$. Since \bar{G} is homomorphic to $\widehat{K}_{2,2}$, if $e_i \in \bar{C}_i$ crosses $e_j \in \bar{C}_j$ in \bar{G} , then without loss of generality, $f(e_i) = 13$ and $f(e_j) = 24$. This shows that \bar{C}_i can only be adjacent to \bar{C}_j in C_\times if \bar{C}_i and \bar{C}_j belong to different partites of C_\times . Thus, we have a proper bipartition of C_\times .

For the converse, assume \bar{G} satisfies Conditions 1, 2, and 3. Since \bar{G} is bipartite, we can 2-color all vertices using colors 1 and 2. Since C_\times is bipartite, we can choose a bipartition $U = \{\bar{C}_{i_1}, \dots, \bar{C}_{i_k}\}$ and $V = \{\bar{C}_{i_{k+1}}, \dots, \bar{C}_{i_m}\}$. Note that by the definition of adjacency in \bar{C}_\times , all pairs of crossing edges have one edge in \bar{G}_U and one in \bar{G}_V .

Re-color the vertices of \bar{G} as follows. In \bar{G}_U change the vertices colored 2 to color 3; in \bar{G}_V change the vertices colored 1 to color 4. Leave the colorings on the non-crossing vertices unchanged. Define $f: \bar{G} \rightarrow \widehat{K}_{2,2}$ so that f takes a vertex of \bar{G} to its color as a vertex of $\widehat{K}_{2,2}$. Note that the vertices originally colored 1 in \bar{G} are now adjacent to vertices colored 2 or 3, while the vertices originally colored 2 in \bar{G} are now adjacent to vertices colored 1 or 4. These colorings match the labels of the edges in $\widehat{K}_{2,2}$. Thus, f preserves adjacencies. Since the edges of \bar{G}_U are colored 13, the edges of \bar{G}_V are

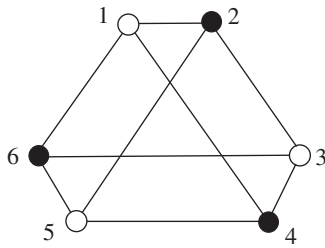


FIGURE 9. $\widehat{K}_{3,3}$.

colored 24, and edges 13 and 24 cross in $\widehat{K}_{2,2}$, f also preserves crossings. Thus, f is a geometric homomorphism. ■

Below we present a simple extension of Theorem 2 to a realization of $K_{3,3}$ in which all crossing components are plane subgraphs. Let $\widehat{K}_{3,3}$ denote the geometric realization of $K_{3,3}$ given in Figure 9. Note that $\theta(\widehat{K}_{3,3})=3$ although the thickness of the abstract graph $K_{3,3}$ is two.

Theorem 3. *A geometric graph \overline{G} is homomorphic to $\widehat{K}_{3,3}$ if and only if*

1. \overline{G} is bipartite;
2. each crossing component \overline{C}_i of \overline{G} is a plane subgraph;
3. $\chi(C_\times) \leq 3$.

Proof. Assume $f: \overline{G} \rightarrow \widehat{K}_{3,3}$. Since \overline{G} is homomorphic to $\widehat{K}_{3,3}$, we get that $\chi(G) \leq \chi(K_{3,3})=2$. Thus \overline{G} is bipartite. Each crossing component of \overline{G} satisfies $f(\overline{C}_i) = 14$ or 25 or 36. Thus, each crossing component is homomorphic to K_2 and is therefore plane. Use the images of the crossing components to partition the vertices of C_\times into three sets, U, V , and W depending on which edge of $\widehat{K}_{3,3}$ each crossing component is sent to by f . It is straightforward to show that this is a proper 3-coloring of C_\times .

For the converse, assume \overline{G} satisfies Conditions 1, 2, and 3. If C_\times is bipartite, then by Theorem 2, \overline{G} is homomorphic to $\widehat{K}_{2,2}$, and there is clearly an injective homomorphism from $\widehat{K}_{2,2}$ to $\widehat{K}_{3,3}$. If $\chi(C_\times)=3$, we can use a proper 3-coloring to partition the crossing components of \overline{G} into three sets, U, V , and W .

Since \overline{G} is bipartite, we can 2-color the vertices using the colors 1 and 2. Re-color the vertices as follows. In \overline{G}_U change vertices colored 2 to color 4; in \overline{G}_V change vertices colored 1 to color 5; in \overline{G}_W change vertices colored 1 to color 3 and vertices colored 2 to color 6. Leave unchanged the colors on the non-crossing vertices of \overline{G} . We can now define f from \overline{G} to $\widehat{K}_{3,3}$ by sending each vertex to its color (as a vertex of $\widehat{K}_{3,3}$). The remainder of the proof is exactly the same as that of Theorem 2. ■

5. GEOCHROMATIC NUMBER FOUR

We now return to the classification of geometric graphs with geochromatic number at most four. Recall that \widehat{K}_4 denotes the thickness-2 geometric realization of K_4 , with $V(\widehat{K}_4) = \{1, 2, 3, 4\}$ and edge 13 crossing edge 24. The geometric graphs in Figure 10

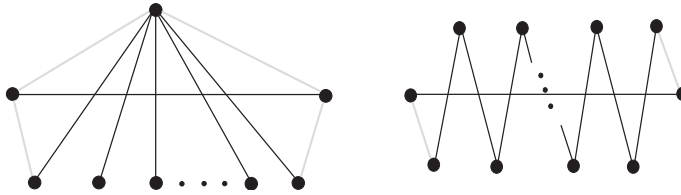


FIGURE 10. 4-geochromatic graphs can have an arbitrary number of crossings.

show that while there is only one crossing in \widehat{K}_4 , there is no bound on the number of crossings in a graph homomorphic to \widehat{K}_4 .

We noted in Section 3 that $X(\overline{G}) \leq 4$ implies $\chi(G) \leq 4$ and $\theta(\overline{G}) \leq 2$. The following theorem provides additional necessary conditions which are similar to those found in Theorems 2 and 3.

Theorem 4. *If \overline{G} is 4-geocolorable, then:*

1. *each crossing component \overline{C}_i is a bipartite plane subgraph of \overline{G} ;*
2. *there is a proper bipartition (V, U) of $V(C_\times)$ so that \overline{G}_U and \overline{G}_V are bipartite plane subgraphs of \overline{G} .*

Proof. Assume \overline{G} is 4-geocolorable. Since there exists a homomorphism $f: \overline{G} \rightarrow \widehat{K}_4$, by Lemma 1 the map from \overline{G}_\times to the crossing subgraph of \widehat{K}_4 is also a geometric homomorphism. Precisely as in the proof of Theorem 2, we can show that each crossing component is bipartite and plane and, using the partition $U = \{\overline{C}_i | f(\overline{C}_i) = 13\}$ and $V = \{\overline{C}_i | f(\overline{C}_i) = 24\}$, that C_\times is bipartite. Appropriate restrictions of the original homomorphism show that \overline{G}_U is homomorphic to the edge 13 and \overline{G}_V is homomorphic to the edge 24, so \overline{G}_U and \overline{G}_V are bipartite plane subgraphs of \overline{G} . ■

Theorem 4 tells us that Conditions 1 and 2 are necessary for a geometric graph to be 4-geocolorable. However, the geometric graph \overline{H} in Figure 7 in Section 4 shows that these conditions are not sufficient. The crossing vertices in \overline{H} are involved in the single crossing and therefore by Observation 2, they cannot be identified. Hence, these four crossing vertices require four colors in a geocoloring. The uncrossed vertex is adjacent to each of the four crossing vertices and so requires a fifth color. Thus $X(\overline{H}) = 5$. We do have the following partial converse to Theorem 4, however.

Theorem 5. *If \overline{G} satisfies Conditions 1 and 2 in Theorem 4, then $\overline{G}[V_\times]$ is 4-geocolorable.*

Proof. Suppose that \overline{G} satisfies Conditions 1 and 2 of Theorem 4. Let U, V be partites for a proper bipartition of $V(C_\times)$ so that each of \overline{G}_U and \overline{G}_V is bipartite and plane. This allows us to properly color \overline{G}_U using colors 1 and 3 and \overline{G}_V using colors 2 and 4. Any edge of $\overline{G}[V_\times]$ that is not in \overline{G}_U or \overline{G}_V has one vertex in each subgraph and thus its vertices are assigned different colors. This is a proper 4-coloring of $\overline{G}[V_\times]$ which defines a mapping $f: \overline{G}[V_\times] \rightarrow \widehat{K}_4$. It is easy to show that f is a geometric homomorphism. ■

The vertex colorings given in Figure 11 show that greed does not always pay when geocoloring. Note that Theorem 5 tells us that $\overline{G}[V_\times] \rightarrow \widehat{K}_4$. Therefore, we can greedily

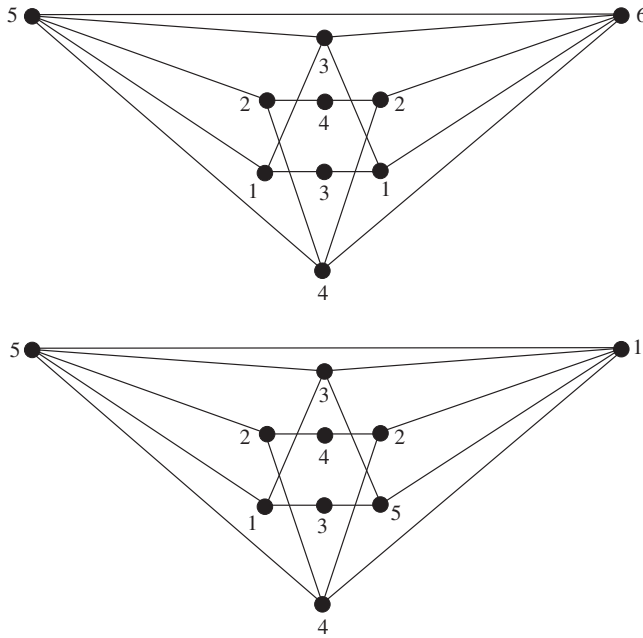


FIGURE 11. Greed does not pay.

geocolor the crossing vertices 1 through 4 accordingly. Up to a permutation of colors, there is only one way to do this. To extend this to a geocoloring of \overline{G} , the two adjacent uncrossed vertices must be colored 5 and 6. This gives $\overline{G} \rightarrow \widehat{K}_6$, where \widehat{K}_6 denotes the convex 6-clique. However, in the second coloring, we geocolor $\overline{G}[V_\times]$ with five colors and then extend the 5-geocoloring to all of \overline{G} . This yields $\overline{G} \rightarrow \widehat{K}_5$, where \widehat{K}_5 denotes the convex 5-clique. Thus, greed in the number of colors used in $\overline{G}[V_\times]$ can yield a less than optimal geocoloring of \overline{G} .

Corollary 5.1. *If \overline{G} satisfies Conditions 1 and 2 in Theorem 4, then $\overline{G} \rightarrow \widehat{K}_4$ if and only if at least one of the 4-geocolorings of $\overline{G}[V_\times]$ given by the construction in the proof of Theorem 5 can be extended to a proper 4-coloring of \overline{G} .*

Corollary 5.2. *If \overline{G} satisfies Conditions 1 and 2 in Theorem 4, then $X(\overline{G}) \leq 8$.*

Proof. Suppose \overline{G} satisfies the Conditions 1 and 2 for being 4-geocolorable given in Theorem 4. Then by Theorem 5, there is a proper vertex coloring of the crossing vertices of \overline{G} with the colors $\{1, 2, 3, 4\}$ so that if two edges cross in \overline{G} , they are colored 13 and 24. The subgraph induced by the non-crossing vertices of \overline{G} is planar by definition. Thus there is a proper 4-coloring of the non-crossing vertices of \overline{G} with the colors $\{5, 6, 7, 8\}$. It is clear that the resulting vertex coloring is a proper 8-coloring. Let \widehat{K}_8 be the convex geometric 8-clique with vertices labeled $1, \dots, 8$ in cyclic order. With this ordering, the edges 13 and 24 cross in \widehat{K}_8 . The map $f: \overline{G} \rightarrow \widehat{K}_8$ implied by this coloring preserves adjacency (since all vertices in \widehat{K}_8 are adjacent) and crossings (since only edges labeled 13 and 24 cross in \overline{G}). Thus, $\overline{G} \rightarrow \widehat{K}_8$ and $X(\overline{G}) \leq 8$. ■

This argument generalizes to geometric graphs with larger geochromatic numbers.

Corollary 5.3. *If $X(\overline{G}[V_\times])=n$, then $X(\overline{G}) \leq n+4$.*

We have given necessary conditions for a graph to be 4-geocolorable. Below we provide sufficient conditions.

Theorem 6. *If \overline{G} satisfies the following conditions, then \overline{G} is 4-geocolorable.*

1. Each \overline{C}_i is a plane subgraph of \overline{G} .
2. There is a proper bipartition of $V(C_\times)$ into partites U and V so that
 - (a) \overline{G}_U and \overline{G}_V are bipartite plane subgraphs, and
 - (b) there exists a set of non-crossing edges $W = \{e_1, \dots, e_m\}$ of \overline{G} , so that each $e_i \in W$ has an identified non-crossing vertex denoted by v_i (called its initial vertex) and vertex w_i (its terminal vertex) satisfying:
 - i. the removal of the edges of W removes all odd cycles from \overline{G} ;
 - ii. no initial vertex v_i is equal to any terminal vertex w_j ;
 - iii. if $v_i = v_j$, then either w_i, w_j are both non-crossing vertices, or both are in \overline{G}_U , or both are in \overline{G}_V .

Proof. Let \overline{G} be as described above. Denote by \overline{G}' the graph we get by removing the edges of W from \overline{G} . Since \overline{G}' contains no odd cycles, it is bipartite. Properly 2-color the vertices of \overline{G}' using the colors 1 and 2.

Since the edges of W are, by definition, incident to non-crossing vertices, these edges are not contained in $\overline{G}[V_\times]$. Thus $\overline{G}[V_\times]$ and its subgraph \overline{G}_\times are subgraphs of \overline{G}' . Thus, they have been properly 2-colored in the bipartite coloring of \overline{G}' . Since \overline{G}_U and \overline{G}_V are subgraphs of $\overline{G}[V_\times]$, they have also been properly 2-colored.

In \overline{G}_U , re-color all vertices colored 1 with color 4; in \overline{G}_V , re-color all vertices colored 2 with color 3. Note that now all vertices of \overline{G}_U are colored 2 or 4 and all vertices of \overline{G}_V are colored 1 or 3. (All non-crossing vertices are colored 1 or 2.) Since each of \overline{G}_U and \overline{G}_V is plane and each pair of crossing edges in \overline{G} has one edge in \overline{G}_U and the other in \overline{G}_V , this gives a 4-geocoloring of $\overline{G}[V_\times]$. Note that since \overline{G} satisfies the conditions of Theorem 6, it satisfies the conditions of Theorem 4. By Corollary 5.1, if we can extend this 4-geocoloring of $\overline{G}[V_\times]$ to a proper 4-coloring of \overline{G} , we will have a 4-geocoloring of \overline{G} .

Our goal is to add the edges of W back into \overline{G}' , one at a time, re-coloring the initial vertices as necessary, until we have a proper 4-coloring of G . Denote \overline{G}' by \overline{G}'_0 and for each $i > 0$, let $\overline{G}'_i = \overline{G}'_{i-1} \cup \{e_i\}$.

We begin with edge e_1 . Recall that v_1, w_1 are the initial and terminal vertices of e_1 and that v_1 is a non-crossing vertex. Since v_1 is non-crossing, it is in \overline{G}' but not in \overline{G}_U or \overline{G}_V . Thus the re-coloring in \overline{G}_U and \overline{G}_V did not affect v_1 , so it is currently colored 1 or 2. Assume the color of v_1 is 1. Then in the original 2-coloring of \overline{G}' , all \overline{G}' -neighbors of v_1 were colored 2. Since vertices colored 2 were either left alone or re-colored 3, the \overline{G}' -neighbors of v_1 can only be colored 2 or 3. If the color of w_1 is not 1, we can leave the color on v_1 alone, add e_1 to \overline{G}' , and have a proper coloring of $\overline{G}'_1 = \overline{G}' \cup \{e_1\}$. If the color of w_1 is 1, re-color v_1 with 4, add in the edge e_1 , and we have a proper 4-coloring of \overline{G}'_1 . Similarly, if the color of v_1 is 2, then its \overline{G}' -neighbors are colored 1

or 4. In this case, if w_1 is not colored 2, we leave the color of v_1 unchanged, and if w_1 is colored 2, we re-color v_1 with 3. In either case, we may add the edge e_1 and obtain a proper 4-coloring of \overline{G}'_1 .

Assume that we have added the edges e_1, \dots, e_{m-1} in such a way that we end up with a proper 4-coloring of \overline{G}'_{m-1} . At each step, either the color on the initial vertex v_i is unchanged, or we changed it from 1 to 4 or from 2 to 3. Recall that by Condition 2(b)ii, no initial vertex is equal to any terminal vertex. Thus in the process of adding in e_1, \dots, e_{m-1} , we never changed a color on a terminal vertex. We next wish to add in e_m . There are two cases to consider.

First suppose that v_m is not equal to any v_j , where $j < m$. Since by Condition 2(b)ii, v_m is also not equal to any w_j , v_m is not incident to any of e_1, \dots, e_{m-1} . Since the only additional adjacencies in \overline{G}'_{m-1} are those provided by e_1, \dots, e_{m-1} , the neighbors of v_m in \overline{G}'_{m-1} are precisely its neighbors in $\overline{G}'_0 = \overline{G}'$. In this case, we can extend the proper 4-coloring on \overline{G}'_{m-1} to a proper 4-coloring on \overline{G}'_m using the same protocol as for v_1 .

Alternatively, suppose that v_m does coincide with one or more previous initial vertices. That is, suppose $v_m = v_{i_1}, \dots, v_{i_r}$, where all $i_j < m$. Assume v_m was originally colored 1 in \overline{G}' ; then the neighbors of v_m in \overline{G}' were originally colored 2. By induction, v_m is currently colored 1 or 4 and its \overline{G}' -neighbors are colored 2 or 3. However, in \overline{G}'_{m-1} , v_m also has neighbors w_{i_1}, \dots, w_{i_r} ; the color on these vertices determines whether we have to re-color v_m .

Assume v_m is currently colored 1. Since we started with a proper 4-coloring of \overline{G}'_{m-1} , none of w_{i_1}, \dots, w_{i_r} is colored 1. Further, by Condition 2(b)iii, $w_{i_1}, \dots, w_{i_r}, w_m$ are either all non-crossing vertices, or they are all in \overline{G}_U , or they are all in \overline{G}_V . Thus the possibilities are:

- w_{i_1}, \dots, w_{i_r} are all non-crossing vertices colored 2;
- w_{i_1}, \dots, w_{i_r} are all in \overline{G}_U and are colored either 2 or 4;
- w_{i_1}, \dots, w_{i_r} are all in \overline{G}_V and are colored 3.

Thus if w_m is not colored 1, we can leave v_m colored 1. If w_m is colored 1, then w_m cannot belong to \overline{G}_U , so neither can any of w_{i_1}, \dots, w_{i_r} . This implies that either all of $w_{i_1}, \dots, w_{i_r}, w_m$ are in \overline{G}_V or all are non-crossing. In this situation, v_m can be re-colored 4, allowing us to add in edge e_m .

Next assume v_m is colored 4 in \overline{G}'_{m-1} . This means that in some previous step, its color was changed because some w_{i_j} is colored 1. Thus, either all $w_{i_1}, \dots, w_{i_r}, w_m$ are in \overline{G}_V or all of them are non-crossing vertices. Either way, none of $w_{i_1}, \dots, w_{i_r}, w_m$ is colored 4, so we can allow v_m to remain colored 4 and add in edge e_m .

A completely analogous argument works if v_m was originally colored 2 in \overline{G}' . Thus, we can extend the 4-geocoloring of $\overline{G}[V_\times]$ to a proper 4-coloring of \overline{G} . By Corollary 5.1, \overline{G} is 4-geocolorable. ■

Example 2. The geometric graph in Figure 12 shows that the conditions in Theorem 6, though sufficient, are not necessary for a graph to be 4-geocolorable. First we will show that \overline{G} is homomorphic to \widehat{K}_4 . We can 4-color the triangulated portion of this geometric graph because it is plane. It is then easy to label the remaining crossing vertices so that pairs of crossing edges are labeled 13 and 24.

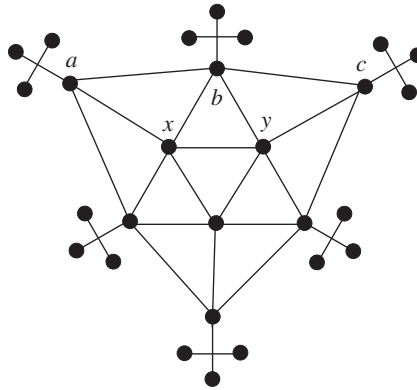


FIGURE 12. $\overline{G} \rightarrow \widehat{K}_4$, but fails the conditions of Theorem 6.

Next, we show that \overline{G} does not meet the conditions of Theorem 6. If a set W as described in Theorem 6 exists, it must contain an edge of the central 3-cycle of \overline{G} . Without loss of generality, we can assume $e_1 = xy \in W$. Also without loss of generality, $x = v_1$ and $y = w_1$. The 3-cycle byc must also have an edge in W , say e_2 . Since e_2 must be incident to a non-crossing vertex, either $e_2 = by$ or $e_2 = cy$. In either case, since v_2 must be a non-crossing vertex $v_2 = y$. But then $w_1 = v_2$, which violates Condition 2(b)ii of Theorem 6.

Corollary 6.1. *If \overline{G} is a bipartite geometric graph, then \overline{G} is 4-geocolorable if and only if \overline{G} satisfies Conditions 1 and 2 of Theorem 4.*

Proof. If $X(\overline{G}) \leq 4$, we get our result by Theorem 4. Conversely, note that Condition 1 of Theorem 4 implies Condition 1 of Theorem 6. Further, if \overline{G} is bipartite and meets Condition 2 of Theorem 4, then since \overline{G} has no odd-length cycles, it meets Condition 2 of Theorem 6 by letting $W = \emptyset$. Hence by Theorem 6, $X(\overline{G}) \leq 4$. ■

6. FUTURE WORK

Any question that is interesting for abstract graph homomorphisms is likely to be interesting for geometric graph homomorphisms, plus a few extra. However, in this section we restrict our attention to questions connected to the current work.

Question 1. By the example in Figure 11, we know there exists a geometric graph that meets the conditions of Theorem 4 and that has geochromatic number 5. Further, we know by Corollary 5.2 that such a geometric graph has geochromatic number at most 8. What is the largest geochromatic number possible in a geometric graph that meets the conditions of Theorem 4?

Question 2. In the proof of Theorem 1, we constructed an infinite family of bipartite, thickness-2 geometric graphs with arbitrarily large geochromatic number. However, each of these graphs contains a large number of crossings. If we restrict the total number of crossings in a geometric graph, can we then use the chromatic number and

thickness to yield, or bound, the geochromatic number? What if we restrict the total number of edges any single edge can cross?

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