Senior Seminar in Philosophical Foundations of Mathematics

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1 Introduction

At Hamilton College, the small liberal arts college at which I teach, every concentration must culminate in a "capstone experience", undertaken by students in their senior year. In the mathematics department, this takes the form of a seminar, capped at 10 students, whose defining characteristic is that class time is devoted almost entirely to student presentation and discussion, with the instructor mostly playing the role of moderator. We offer seminars in a number of different areas; I developed one in 2008 focusing on the famous set paradoxes, that I somewhat grandly titled Philosophical Foundations of Mathematics. Since then, it has been offered four times, and each time, I have received a very positive reaction from students, who tell me that they appreciated not only the new mathematical content, but also the opportunity to discuss broader nature of the discipline as a human intellectual endeavor.

2 Description of the Course

To recruit students, I circulate the following to juniors pondering their senior seminar options.

In the first few weeks of real analysis, you learned that there are different types of infinite sets, namely countable and uncountable sets. Can we treat the cardinalities of infinite sets as actual numbers? Georg Cantor thought so, and in the late 19-th century, he developed an extensive theory of what he called transfinite numbers, including

- *how to tell when two transfinite numbers are different;*
- *how to compare the sizes of two different transfinite numbers;*
- *how to add, multiply and exponentiate transfinite numbers;*
- *how many different transfinite numbers there are.*

At first, Cantor's work met with considerable resistance from the mathematical community, but by the turn of the century, the world's foremost mathematician, David Hilbert, declared, "no one will drive us from the paradise that Cantor has created for us"! Then, disaster struck. Various mathematicians, including Cantor himself, started finding troublesome paradoxes in the theory of infinite sets. Serious mayhem in the foundations of mathematics ensued. Many issues remain unresolved to this day.

The seminar meets for two 75-minute classes a week, and is divided into two quite different parts. During the first nine and half weeks of the semester, students learn the basics of naive and axiomatic set theory. The last four and half weeks of the course are devoted to reading and discussing a selection of papers on some of the philosophical issues raised by this material.

2.1 Set Theory

I use the Moore method for the first portion of the course, inspired by my colleague Richard Bedient, who has been using it for years in his topology senior seminar. (See his recent article in PRIMUS [1].) Students are collectively responsible for fleshing out the skeleton of a textbook on set theory; it contains definitions as well as lemmas, propositions and theorems, but the students must supply examples and proofs. In writing this skeletal textbook, I borrowed very heavily from Seymour Lipschutz's volume in the *Schaum's Outlines* series [7] for material on naive set theory, and from Robert Wolf's *A Tour Through Mathematical Logic* [12] and Mary Tiles' *The Philosophy of Set Theory: An Historical Introduction to Cantor's Paradise* [11] for axiomatic set theory. A list of the chapter titles indicates the topics covered.

1. Naive Set Theory: Basic Definitions

2. Cardinal Numbers

3. Ordered Sets

4. Well-Ordered Sets

5. Ordinal Numbers

6. Set Paradoxes

7. Axiomatic Set Theory

8. Transfinite Numbers in Zermelo-Fraenkel Set Theory

9. The Cumulative Hierarchy of Sets

As a sample of what the students are given, please see the current version of Chapter 1, in the appendix to this paper. We cover this in the first 75-minute class meeting. Tex files for all chapters are available on my home page (people.hamilton/edu/scockbur). An additional facet of this course is that students learn how to typeset mathematics using TEX; we usually refer to the textbook as the TEXbook.

Students are expected to come to class having read and 'filled in' the shaded boxes of the section(s) assigned for that day; they hand in their work in at the end of the class. During class time, students take turns presenting their examples/proofs at the board; the audience is expected to respond actively, by constructively pointing out errors, asking for clarification and/or offering alternative approaches. This feedback is also directed to me; every year, students have made suggestions for reorganizing, adding to or deleting from the TEXbook to improve it for future incarnations of the seminar. Each class, a different student acts as the scribe, responsible for recording the presented material and inserting it in the appropriate sections of the textbook.

For this portion of the course, I grade students not only on the homework they submit, but also on their willingness to present their work at the board and to respond to their fellow students' presentations. Students tend to be quite shy initially, but I do my best to engender an atmosphere that is informal and collaborative, rather than competitive. Once they see that everyone makes mistakes at the board, even the 'math stars', they relax and eventually are jostling to present what they see as the choice proofs. I give two midterms on this material, because that is the incentive that math majors are accustomed to for keeping up with terminology and theory.

As students work through the TEXbook, they experience increasing disorientation. Most have taken real analysis with me as juniors, and after some resistance eventually made their peace with the distinction between countably and uncountably infinite sets. They get a thrill from learning how Cantor defined addition, multiplication and exponentiation of his transfinite cardinal numbers, but tend to view this work as they would fantastical stories of fictional characters. (Tellingly, the fall 2012 seminar students designed a cover for the TEXbook featuring characters from the recent film adaptation of J. R. R. Tolkien's *The Hobbit*, and inserted as an epigram the following quote from J. K. Rowling's *Harry Potter and the Deathly Hollows*; "Of course it is happening inside your head, Harry, but why on earth should that mean it is not real?" [9]) This prompts discussion on the ontological status of infinite numbers. I prod them on their opinion on the ontological status of finite numbers, or indeed triangles, functions, vector spaces and other mathematical objects. This leads naturally to the question of whether mathematics is somehow out there in the real world (the Platonist position), or is simply a social construction. However, I limit this discussion to about ten minutes or so, saying that we will return to this debate in the second portion of the course. My aim at this point is merely to raise philosophical issues, while maintaining as neutral a stance as possible. I believe students need some time to think through their own positions, and the opportunity to reevaluate them as we work through the technical material.

A theorem students are asked to prove in Chapter 5 asserts that every ordinal number has an immediate successor. I can usually count on the person presenting this proof to argue that given any well-ordered set A, we can let x be something that is not in A, and then $\{A : \{x\}\}$ will be a well-ordered set of greater ordinality. I can also usually count on someone in the class asking, "What if A is the set of everything (i.e. the universal set)?" Again, I suppress debate after about ten minutes of discussion, essentially sweeping the problem under the rug. Students start to sense that we are rowing a boat that is springing leaks, replicating the growing unease the mathematical community of the late nineteenth century experienced. Serendipitously, we cover the famous set paradoxes the day before the October fall recess, so students have a long weekend to ponder the implications. Until this point, most of these math majors, either consciously or unconsciously, have viewed their subject as a bastion of Absolute Truth. They have never had occasion to call into question an entire body of material, that had seemingly been proved using the same trusted logical techniques they have seen throughout their college careers.

When students get back to campus and we progress to axiomatic set theory, they tend be suspicious of the entire enterprise. The Axioms of Foundation and Replacement seem far from "self-evident". The Subset Axiom and Power Set Axiom seem to be included only to preserve Cantor's theorem, which, to many students' minds, seems to be the main source of the problem. The distinction between Zermelo-Fraenkel sets and classes seems artificial. They bristle at the line in the TEXbook that states, "like the primitive terms and axioms, the laws of logic are simply accepted without explanation or proof of validity". As we work though the chapter on transfinite numbers in ZF, students are able to prove propositions, but struggle to gain an intuitive understanding of transitive sets, let alone ordinals and cardinals; I believe this paves the road for their appreciation the tenets of formalism. The final chapter of the TEXbook covers the cumulative hierarchy of sets; to help prepare students for this material, I have them read George Boolos's essay "The iterative concept of set" [2]. This serves as a nice transition to the second portion of the course.

2.2 Readings

For this portion of the course, students are required to come to class with a one-page response to the day's reading(s) that includes a concise summary of the main points as well as a list of questions or issues they want to discuss. I appoint someone to start off the class by presenting their response for the first 15 or 20 minutes; the remainder of the class is a free-wheeling discussion on whatever students found most intriguing or controversial. I collect their one-page responses at the end of the class and assign them a letter grade; I also grade them on their participation in class discussion.

By this point, students are bursting to engage in the philosophical debates that have been percolating under the surface for nine weeks. Before getting to papers dealing directly with the set paradoxes, however, I assign three readings providing some historical context. We spend one class each on:

1. A. W. Moore, *The Infinite*; Introduction, Chapters 1 & 2 [8]

2. G. Berkeley, "The Analyst" [3]

3. I. Kant, *Critique of Pure Reason*; Introduction, Transcendental Aesthetic, sections 1 - 7 [6]

The selections from Moore's book summarize classical Greek approaches to the infinite, including a summary of Xeno's paradoxes, a comparison between metaphysical and mathematical infinity, and Aristotle's distinction between actual and potential infinity. I include Berkeley's paper so that we can discuss how a prior crisis in mathematics, also involving the infinite, was resolved by replacing the notion of infinitesimals with the notion of limits (essentially, replacing a concept based on actual infinity with one based on potential infinity). The Kant reading introduces students to his classification of knowledge as *a priori* or *a posteriori*, analytic or synthetic. We also compare Kant's views on the relative merits of mathematics and metaphysics with those of Berkeley. This usually prompts a general discussion of faith in mathematics, particularly faith in axioms and faith in the laws of logical inference.

The readings for the next class are:

4. (a) G. Cantor, "Grundlagen" sections 1 - 8; Late Correspondence with Dedekind and Hilbert [4]

   (b) D. Hilbert, "On the infinite" [2]

In section 8 of "Grundlagen", Cantor draws a distinction between what he calls the immanent and transient reality of numbers, which allows students to return to a discussion of the ontological status of numbers. In this section, Cantor also makes a poignant argument for intellectual freedom; students are surprised to learn that issues of censorship arise in so apparently objective a discipline as mathematics. Cantor also writes in this work that new mathematical concepts should be judged on their fruitfulness, meaning presumably their success at generating new theorems; students are surprised to learn that such a market-driven approach could apply to their field. Hilbert's seminal paper serves as a linchpin for the course; in it, Hilbert decries the chaos produced by the set paradoxes, sets as a goal the preservation of as much of Cantor's approach to transfinite numbers as possible, lays out the basic principles of formalism and describes his famous program.

The next class is devoted to examining the schools of logicism, formalism and intuitionism; the readings I assign are:

5. (a) A. Heyting, "Disputation" [2]

   (b) B. Russell, "Introduction to Mathematical Philosophy" [2]

   (c) L. E. J. Brouwer, "Intuitionism and formalism" [2]

I include the amusing paper by Heyting because most students find Brouwer's more formal paper difficult to understand. Student reactions to intuitionism vary greatly; some years I am the only one who is willing to defend the intuititionist approach, other years I am the only one who is wiling to criticize it.

At this point in the fall semester, we have reached our week-long Thanksgiving break. In preparation for our coverage of Gödel, I assign Rebecca Goldstein's highly readable book *Incompleteness: The Proof and Paradox of Kurt Gödel* [5] over the holiday. I treat this a background material only; I tell students we will not spend any class time discussing the book. To my surprise, they still read it. Our week on Gödel is as follows:

6. K. Gödel, "What is Cantor's continuum problem?" [2]

7. *Gödel's Incompleteness Theorems* (worksheet available at people.hamilton/edu/scockbur)

The paper by Gödel includes his eloquent defense of Platonism; after stating his belief that the Continuum Hypothesis must be either true or false, he muses on the likelihood of it being one or the other. Students are surprised that he refers to the fruitfulness criterion mentioned by Cantor; I have to repeatedly remind them that Gödel is only using it as one tool in making an educated guess. There is obviously not enough time in the course to give a very thorough treatment of Gödel's incompleteness theorems, but my worksheet goes into somewhat more detail than Goldstein's book. This is one instance where I stray from the seminar format and give a traditional lecture; the only other instances are lectures on the Cantor-Schröder-Bernstein Theorem and the Well-Ordering Theorem.

For the last week of the semester, the readings are:

8. A. J. Ayer, "The a priori" [2]

9. Max Tegmark, "The Mathematical Universe" [10]

Goldstein's book tends to demonize logical positivism, so I believe it is important to assign a reading by someone who can defend that position. In some incarnations of the course, I have used Carnap's *Empriricism, semantics and ontology* [2], Quine's *Two Dogmas of Empiricism* or extracts from Wittgenstein's *Tractatus* (both available online) for this purpose, but I have found students more receptive to Ayer's arguments. Discussion often centers around Ayer's contention that Kant provides both semantic and pyschological criteria for distinguishing between analytic and synthetic knowledge. Ayer argues that a sufficiently intelligent being would 'know' all of mathematics immediately upon being informed of definitions of the terms involved; mathematics is simply one vast tautology that tells us nothing about the empirical world. Max Tegmark's article provides a sharp contrast; its thesis is that the universe *is* a mathematical structure.

At the end of this portion of the course, students write a 6 - 8 page final paper. The assignment is to select a paper in the philosophy of mathematics that is not covered in class and (i) summarize the main points of the paper, (ii) put the paper in the context of the material we have read, and (iii) explain their own areas of agreement and disagreement with the paper. Roughly a week before the paper is due, I have the class over to my house for dinner, where they must each give a 10-minute informal oral presentation on the work they have done on their paper so far. I have found this step invaluable in helping students focus their ideas and anticipate possible counterarguments to their positions.

3 Student Reaction

The best student evaluations I have received have been from students in this course.

[T]his is the best course I have taken to Hamilton to date. The combination of technical mathematics, introduction to respected mathematical writing, and discussion of mathematical philosophy gave rise to an enormous development in thinking skills and an enormous expansion of our (students) knowledge of set theory at a much higher level than I would have expected in the undergraduate curriculum.

I thought the class was great. I really enjoyed both parts of the course: the TEXbook and reading philosophical papers. I thought the two halves went very well together.

This is the most intellectually challenging class I have ever taken. I found it very challenging at first to write proofs on the board in front of an audience of peers, but I felt that I acclimated to this environment later in the semester. Class discussion were extremely engaging.

The seminar style, student based discussion and lecture is an excellent way to learn mathematics. Hearing other students present proofs and their method of thinking is invaluable to developing one's own mathematical knowledge. The discussion in the philosophy third of the class was extraordinarily interesting, engaging and intellectual.

The course was extremely interesting and challenged everyone's opinions and views. I cannot imagine a better way to stretch myself academically. As a primarily student run course, everyone in the class took pride in the work they did. It was the most open and rigorous learning environment I've ever been in.

I thought this class was exceptional — some really mind-bending material that probed at the very heart of mathematics.

It is clear from these comments that the success of the course owes as much to the seminar format as to the content. Because the class size is small (twice there have been 10 students, twice there have been only 5 students), a lot of bonding goes on, both inside and outside the classroom. Another major factor in the success of the course, in my opinion, is the presence of a large number of students pursuing double majors. For example, students who are also concentrating in history have provided valuable insights on the intellectual environments in which Aristotle, Berkeley and Kant worked; in particular, they caution their classmates against the arrogance of dismissing thinkers who were unaware of, say, non-Euclidean geometry. Students also concentrating in psychology and neuroscience tell others in the class about experiments that suggest that animals such as pigeons and rats subitize (demonstrate primitive arithmetical abilities); this enriches our discussions about whether mathematics exists inside or outside human beings. Students also concentrating in physics contribute their knowledge of quantum theory and astrophysics to our discussions of whether infinity exists in the physical world; they can contribute a modern response to Berkeley's query "whether there be any need of considering Quantities either infinitely great or infinitely small?" [3] Double concentrators in computer science relate Gödel's work on incompleteness to computability and the halting problem.

4 Final Thoughts

I believe that this course gives students the opportunity to step outside the inner workings of mathematics and get some sense of the discipline as a whole. As senior math concentrators, they have been exposed to a number of courses in pure and applied mathematics, and as liberal arts students, they have been trained in critical thinking as well as written and oral communication skills. In the foundations seminar, they can bring all of their intellectual tools to the table and forge a memorable capstone experience.

5 Appendix

Naive Set Theory: Basic Definitions

Definition. A *set* is a collection of objects, called the *elements* or *members* of the set.

Note. For now, the terms 'set' and 'collection' will be treated as synonymous.

The empty set is a collection of no objects, denoted either $\emptyset$ or $\{\}$. (Be careful: $\{\emptyset\}$ denotes....)

Definition. Set B is a *subset* of set A iff ..... The *power set* of a set A, denoted $\wp(A)$, is....

Note. To show that two sets A and B are equal, ...

Definition. The *union* of sets A and B is given by $A \cup B =$..., and the *intersection* is $A \cap B =$... The *relative complement* of B in A is $A \backslash B =$...

Definition. The *Cartesian product* of two sets A and B is given by

$$A \times B = \{(a, b) \,|\, a \in A, b \in B\}.$$

More generally, for a finite collection of sets $A_1, A_2, \ldots, A_n$,

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n)\} \,|\, a_i \in A_i\}.$$

Definition. If A and B are sets, a *relation from A to B* is a subset R of $A \times B$.

Notation. For all $a \in A, b \in B$, $a\,R\,b \iff (a, b) \in R$.

Example 5.1. Let $A = \{1, 2, 3, 4\}$ and $B = \{10, 11, 12, 13\}$ and define a relation by $a\,D\,b \iff a$ divides b. Then $D =$...

Definition. An *equivalence relation* is a relation R from a set S to itself that is reflexive, symmetric and transitive; that is, ...

Definition. If R is an equivalence relation on a set S and $x \in S$, then *the equivalence class of x in S under R* is given by $[x] = \{y \in S \,|\, (x, y) \in R\} = \{y \in S \,|\, x\,R\,y\}$.

Example 5.2. Let $S = \mathbb{Z}$, and define $x\,R\,y \iff x - y$ is divisible by 3; that is, $x - y = 3k$ for some $k \in \mathbb{Z}$. Then R is an equivalence relation on $\mathbb{Z}$ because ... . Moreover, $[0] =$ ....

Theorem 5.1. *An equivalence relation R partitions a set S into equivalence classes.*

Proof. We must show that every $x \in S$ belongs to one and only one equivalence class. Since R is reflexive, $x \in [x]$. Now suppose that we also have $x \in [y]$ for some $y \in S$. Then we must show that $[x] = [y]$. ..... □

Definition. A *function* $f : A \to B$ is a relation $f \subseteq A \times B$ such that for each $a \in A$, there exists exactly one ordered pair in f whose first coordinate is a. An *injection* or *one-to-one* function is...; a *surjection* or *onto* function is....; and a *bijection* is.... [Phrase these in terms of this new definition of a function as a set of ordered pairs.]

Note. If $A \neq \emptyset$, [are there any functions $A \to \emptyset$ or $\emptyset \to A$? What about $\emptyset \to \emptyset$?]

Notation. If $f : A \to B$ is a function, then for all $a \in A$ and $b \in B$, $f(a) = b \iff (a, b) \in f$.

Definition. If $f : A \to B$ is a function and $C \subset A$, then the *restriction of* f *to* C is the function $g : C \to B$ defined by $g(c) = f(c)$ for all $c \in C$.

Notation. The restriction of f to C is sometimes denoted $f|_C$.

Note. Any restriction of an injective function is injective. This is not the case for surjectivity.

Note. If $f : A \to B$ is a function, then $f : A \to f(A)$ will be a surjective function.

Definition. Let $\{A_i \,|\, i \in I\}$ be an arbitrary collection of sets. Then the union and intersection of the collection are respectively

$$\bigcup\{A_i \,|\, i \in I\} = \{x \,|\, x \in A_i \text{ for some } i \in I\} \text{ and } \bigcap\{A_i \,|\, i \in I\} = \{x \,|\, x \in A_i \text{ for all } i \in I\}.$$

Definition. Let $\{A_i \,|\, i \in I\}$ be a nonempty collection of nonempty sets; that is, the index set I could be finite, countably infinite or uncountable. A *choice function* is any function $f : \{A_i \,|\, i \in I\} \to \bigcup\{A_i \,|\, i \in I\}$ satisfying $f(A_i) \in A_i$. That is, a choice function 'chooses' exactly one element from each set in the collection.

Definition. The *Cartesian product* of a nonempty collection of nonempty sets $\{A_i \,|i \in I\}$, denoted by $\prod\{A_i \,|\, i \in I\}$, is the set of all choice functions defined on $\{A_i \,|i \in I\}$.

Note. This generalizes the definition of Cartesian products of finitely many sets because...

AXIOM OF CHOICE. *The Cartesian product of a nonempty collection of nonempty sets is nonempty. Equivalently, there exists a choice function for any nonempty collection of nonempty sets.*

Note. This is an *axiom*, not a theorem. We do not *prove* it is true, we *accept* it as true.

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