The Homomorphism Poset of $K_{3,3}$

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Abstract

A *geometric graph* $\overline{G}$ is a simple graph drawn in the plane, on points in general position, with straight-line edges. We call $\overline{G}$ a *geometric realization* of the underlying abstract graph G. A *geometric homomorphism* $f : \overline{G} \to \overline{H}$ is a vertex map that preserves adjacencies and crossings (but not necessarily non-adjacencies or non-crossings). Geometric homomorphisms can be used to define a partial order on the set of isomorphism classes of geometric realizations of an abstract graph G. In this paper, the homomorphism poset of $K_{3,3}$ is determined.

1 Geometric Realizations

In [3], Harborth defines a *good* drawing of a graph G to be a drawing in the plane in which any two edges intersect at most once, and no three edges intersect at a common point; however, edges need not be represented with straight lines. He further defines any two such drawings to be isomorphic if and only if there exists a graph isomorphism that preserves edge crossings and non-crossings, as well as regions and parts of edges. By these definitions, there are 102 non-isomorphic good drawings of $K_{3,3}$. Harborth also proves in this paper that if $m \equiv n \equiv 1$ mod 2, then the parity of the number of crossings in any good drawing of $K_{m,n}$ is the same.

A *geometric realization* (or *rectilinear drawing*) of a graph G is a drawing in the plane in which vertices are in general position and all edges are represented by straight lines. Two realizations of G are isomorphic if and only if there exists a graph isomorphism that preserves edge crossings and non-crossings; this is weaker concept than Harborth's because it does not take into account regions and parts of edges.

To determine the number of non-isomorphic geometric realizations of $K_{3,3}$, note that any geometric realization of $K_{3,3}$ can be completed to obtain a geometric realization of K_6. As shown in [2], there are 15 different geometric realizations of K_6, and for each of these, there are 10 ways of dividing the labeled vertices into two partite sets, but not all of these will result in different realizations of $K_{3,3}$. The 19 different geometric realizations of $K_{3,3}$ are given in Figure 1. Since any geometric realization is a good drawing, Harborth's result also explains why the number of crossings is always odd.

To demonstrate that these realizations are non-isomorphic, we make use of some results from [2]. First we recall some geometric graph invariants.

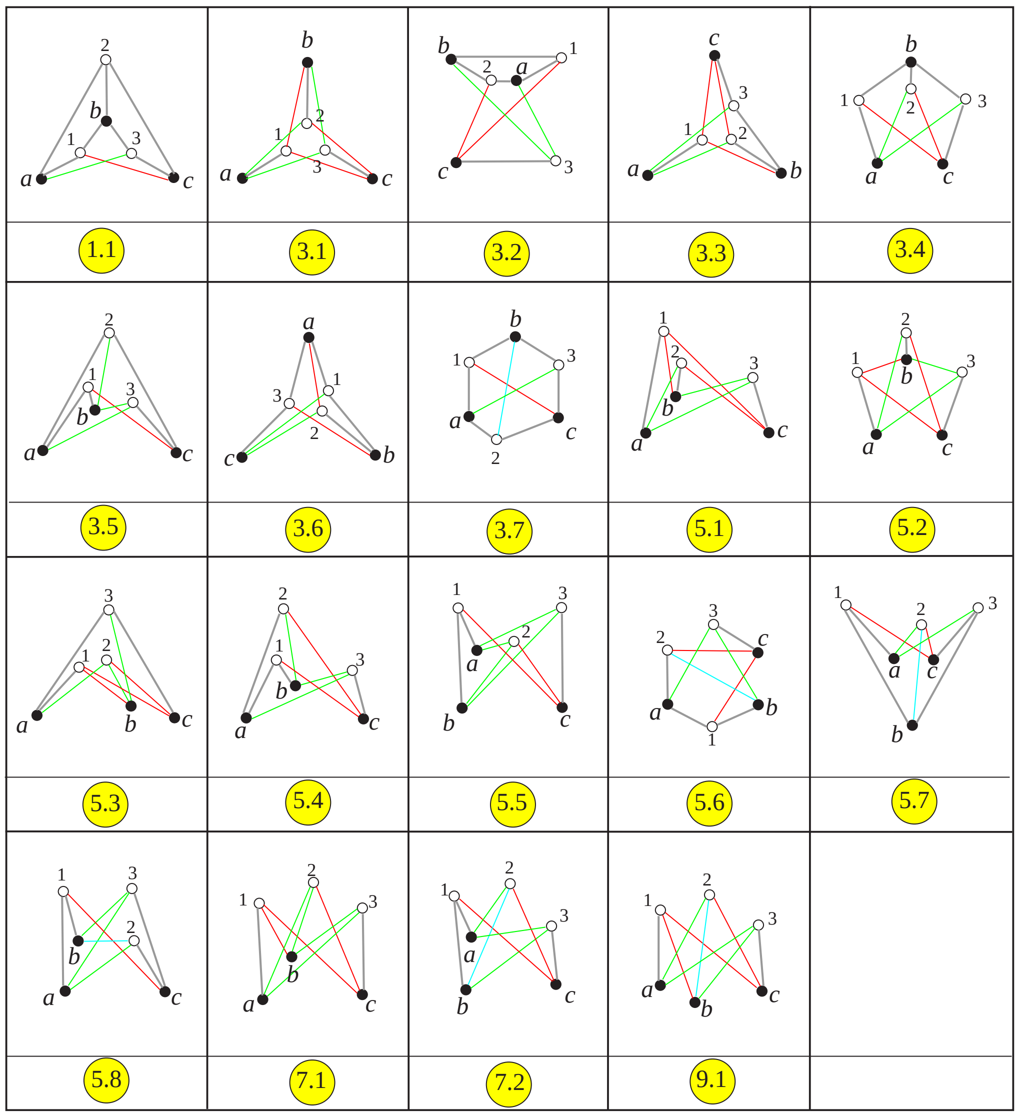

Figure 1: The nineteen geometric realizations of $K_{3,3}$.

Definition 1. Let $\overline{G}$ be a geometric realization of a graph G. Then

1. $cr(\overline{G})$ is the total number of edge crossings in $\overline{G}$;
2. for all $e \in E(\overline{G})$, $cr(e)$ is the number of edges crossing e in $\overline{G}$;
3. $E_0 = \{e \in E(\overline{G}) \mid cr(e) = 0\}$ is the set of uncrossed edges;
4. the *uncrossed subgraph* of $\overline{G}$ is the abstract graph $\overline{G}_0 = (V(G), E_0)$;
5. the *edge crossing graph* of $\overline{G}$ is the abstract graph $EX(\overline{G})$ whose vertices are the edges of G, with adjacency when the corresponding edges of $\overline{G}$ cross;
6. the *line/crossing graph* of $\overline{G}$ is the 2-edge colored abstract graph $LEX(\overline{G})$ whose vertices are the edges of G, with solid edges corresponding to the edges of $EX(\overline{G})$ and dashed edges corresponding to the edges of the line graph, $L(G)$ (indicating when two edges of G are adjacent).

Figure 2 gives the uncrossed subgraphs and Figure 3 the line/crossing graphs of the realizations in Figure 1.

It suffices to show that realizations with the same number of edge crossings are not isomorphic. From Figure 2, note that all realizations of $K_{3,3}$ with 3 crossings have different uncrossed subgraphs, except for realizations 3.5 and 3.6, both of which have uncrossed subgraph P_6. However, the edge crossing graph of realization 3.5 has a vertex of degree 3, while that of 3.6 has maximum degree 2.

Moving on to realizations of $K_{3,3}$ with 5 crossings, realizations 5.1 and 5.2 both have $3K_2$ as uncrossed subgraph, but 5.1 has edge crossing graph $C_4 \cup K_2 \cup 3K_1$, whereas 5.2 has edge crossing graph $P_6 \cup 3K_1$. Realizations 5.4 and 5.6 both have $P_4 \cup K_2$ as uncrossed subgraph, but the edge crossing graph of the former is a tree and of the latter is a 5-cycle.

2 Poset Structure

Geometric homomorphisms were introduced in [1] as a natural generalization of abstract graph homomorphisms. A *geometric homomorphism* $f : \overline{G} \to \overline{H}$ is a vertex map that preserves adjacencies and crossings, but not necessarily non-adjacencies or non-crossings. If $\overline{G}$ and $\widehat{G}$ are geometric realizations of G, set $\overline{G} \preceq \widehat{G}$ if and only if there is a vertex-injective geometric homomorphism $f : \overline{G} \to \widehat{G}$. It is easy to verify that this defines a partial order on the set of all isomorphism classes of geometric realizations of G; the resulting structure is called the it homomorphism poset of G, denoted $\mathcal{G}$.

The Hasse diagram for the poset $\mathcal{K}_{3,3}$ is given in Figure 4. The nodes with blue circumferences correspond to realizations of edge thickness 3.

In [2], it is shown that the homomorphism posets of other graphs of order $n = 6$, namely P_6, C_6 and K_6, are neither lattices nor graded posets. However, it is obvious from Figure 4 that $\mathcal{K}_{3,3}$ is a graded poset with rank function $\rho(\overline{G}) =$

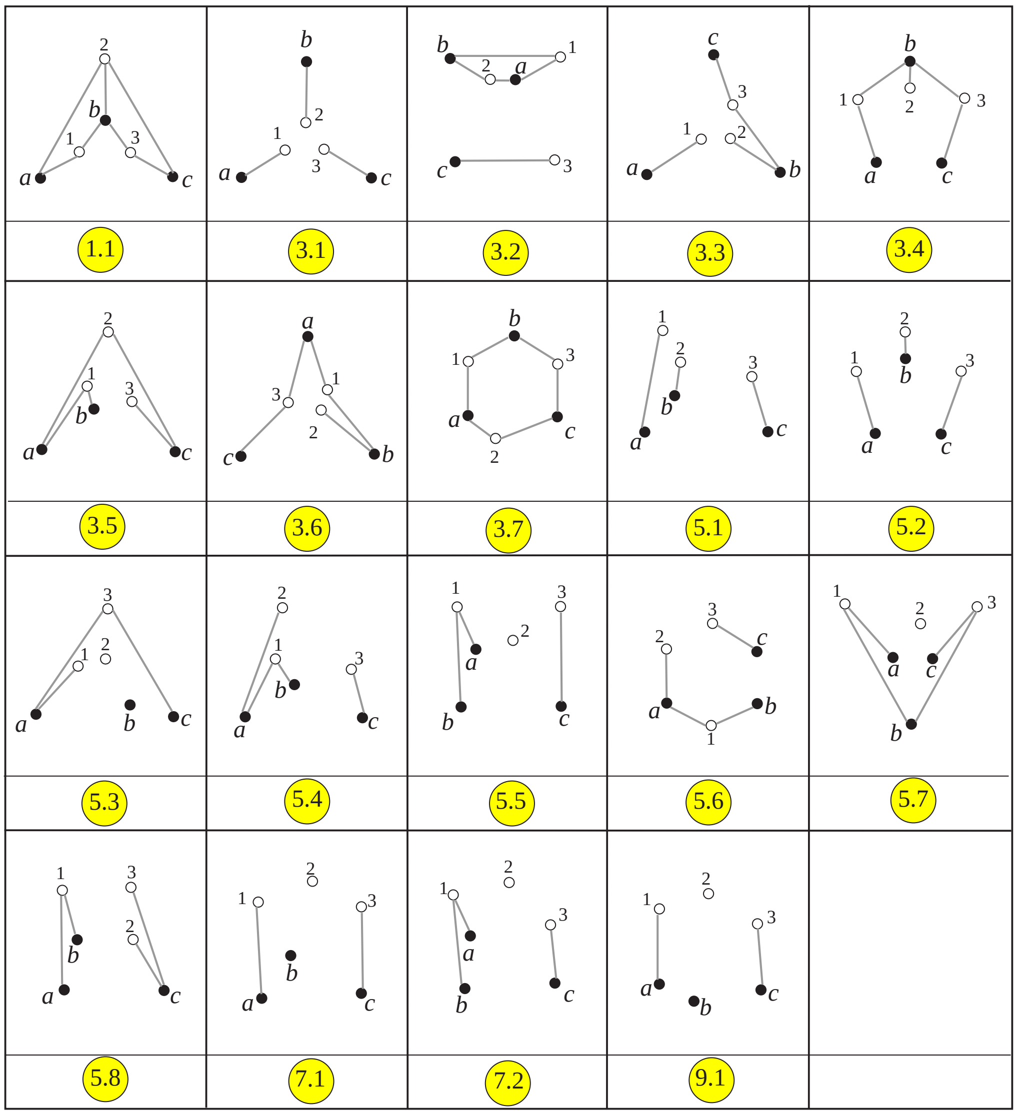

Figure 2: The uncrossed subgraphs of the realizations in Figure 1.

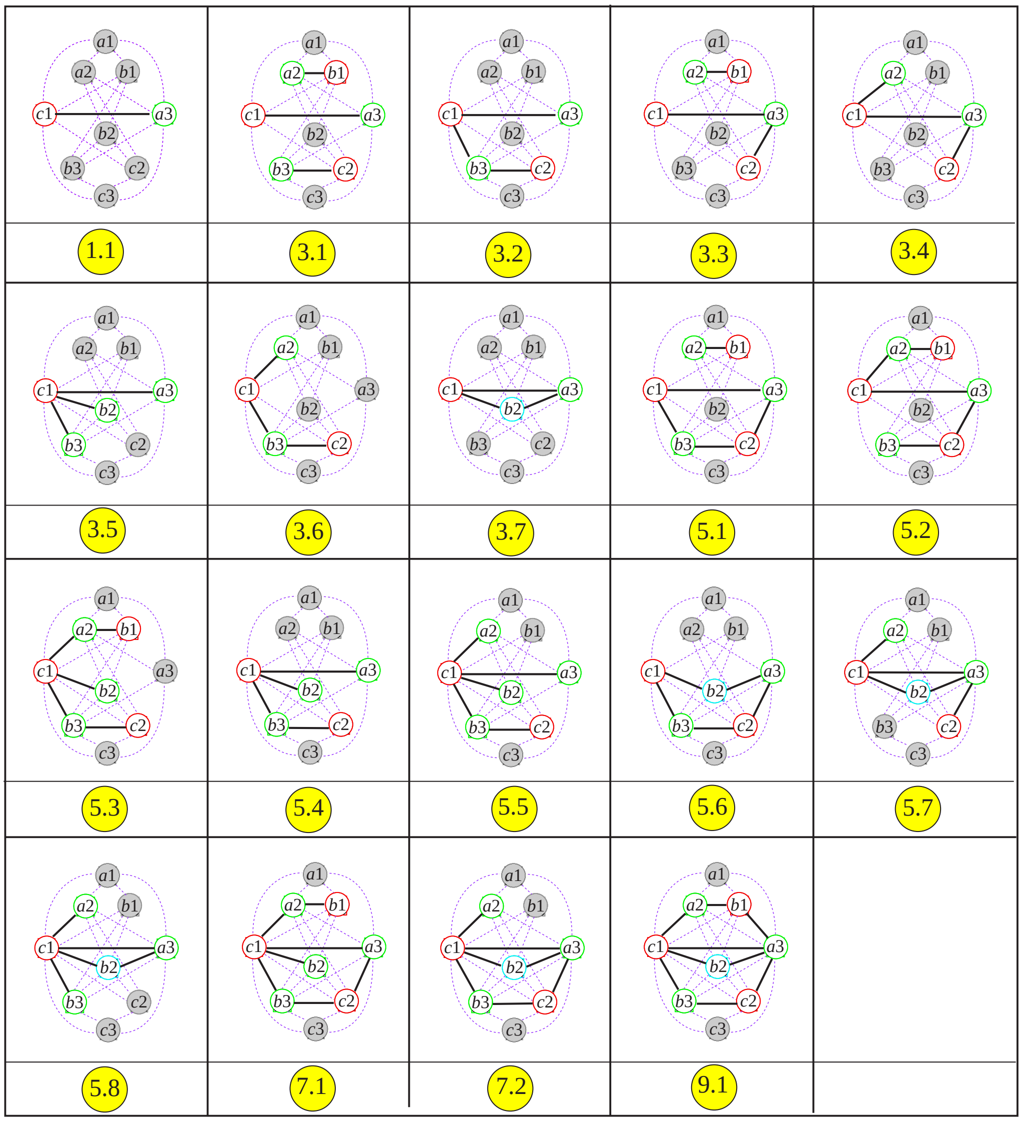

Figure 3: The line/crossing graphs of the realizations in Figure 1.

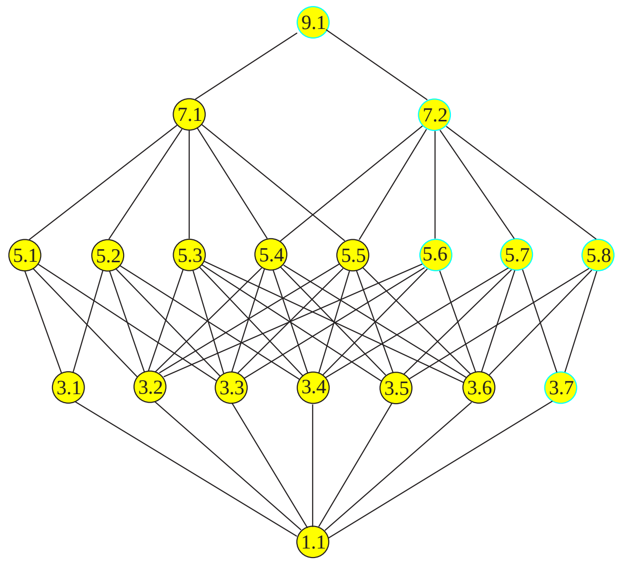

Figure 4: The Hasse diagram of $\mathcal{K}_{3,3}$.

$\lfloor cr(\overline{G})/2 \rfloor$. However, $\mathcal{K}_{3,3}$ is not a lattice; for example, realizations 3.1 and 3.2 have both realizations 5.1 and 5.2 as suprema. Since the poset has a unique maximum, every geometric graph that is homomorphic to some realization of $K_{3,3}$ is homomorphic to realization 9.1; note also that every geometric graph that is homomorphic to a realization of $K_{3,3}$ of edge thickness 2 is homomorphic to realization 7.1.

Most edges in the Hasse diagram of $\mathcal{K}_{3,3}$ are induced by the identity map. The only ones that aren't are from realizations with 3 edge crossings to those with 5 edge crossings. Details are given in Table 1 and Table 2.

Table 1: Non-identity isomorphisms on $K_{3,3}$.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
a	b	c	c	b	a	b	3
b	a	a	b	a	b	a	1
c	c	b	a	c	c	c	2
1	2	3	3	1	1	1	b
2	1	2	2	2	3	3	a
3	3	1	1	3	2	2	c

Missing edges in the Hasse diagram of $\mathcal{K}_{3,3}$ can be justified by appealing to the following result from [2].

Table 2: Geometric homomorphisms from 3-crossing to 5-crossing realizations.

realization	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8
3.1	id	id						
3.2	id	f_1	f_1	id	id	id		
3.3	id	id	f_1	f_2	f_3	f_2		
3.4		id	f_4	f_4	f_4	f_4	id	
3.5			f_5	id	id		f_6	id
3.6			id	f_4	id	id	f_4	f_7
3.7							id	id

Proposition 1. *[2] Let $\overline{G}$ and $\widehat{G}$ be geometric realizations of a graph G, and suppose $\overline{G} \overset{f}{\preceq} \widehat{G}$. Then each of the following conditions holds.*

1. *$\widehat{G}_0$ is a subgraph of $\overline{G}_0$.*
2. *f induces a graph homomorphism $EX(\overline{G}) \to EX(\widehat{G})$;*
3. *f induces a color-preserving graph homomorphism $LEX(\overline{G}) \to LEX(\widehat{G})$ that restricts to an automorphism on $L(G)$.*

Part (1) and Figure 2 together show that:

- realization 3.1 does not precede realizations 5.3, 5.4, 5.4, 5.5, 5.6, 5.7 and 5.8;
- neither realization 3.2 nor realization 3.3 precedes realizations 5.7 and 5.8;
- realizations 5.1, 5.2 and 5.3 do not precede realization 7.2.

Part (2) and Figure 3 together show that:

- realization 3.5 does not precede realizations 5.1, 5.2 or 5.6;
- realization 3.7 does not precede realizations 5.1, 5.2, 5.3, 5.4, 5.5 or 5.6;
- realizations 5.6, 5.7 and 5.8 do not precede realization 7.1.

Part (3) and Figure 3 together show that:

- realization 3.4 does not precede realization 5.1;
- realization 3.4 does not precede realization 5.8;
- realization 3.6 dos not precede realizations 5.1 and 5.2.

References

[1] Debra Boutin and Sally Cockburn. Geometric graph homomorphisms. *Journal of Graph Theory*, 69(2):97–113, February 2012.

[2] Debra Boutin, Sally Cockburn, Alice Dean, and Andrei Margea. Posets of geometric graphs. *Ars Mathematica Contemporanea*, 5:265–284, 2012.

[3] Heiko Harborth. Parity of number of crossings for complete n-partite graphs. *Mathematica Slovaca*, 26:77–95, 1976.