

## A Counter-example to the Admissibility of the $\gamma$ -Filtration on 2-Groups\*

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### INTRODUCTION

In 1961, M. F. Atiyah described a spectral sequence linking the integral cohomology of a finite group  $G$  to a graded version of its complex representation ring, compatible with product structures and restriction and transfer maps,

$$H^*(G, \mathbb{Z}) \Rightarrow Gr_* R(G).$$

The grading is based on the topological filtration of  $R(G)$ , which is in turn derived from the skeletally induced filtration on the  $K$ -theory of the classifying space,  $BG$ . Atiyah conjectured that the Grothendieck, or  $\gamma$ -filtration provided a purely algebraic description of this filtration, but this has since been shown to be false. However, in 1983, C. B. Thomas showed that the topological filtration would nonetheless be algebraically determined by the  $\gamma$ -filtration provided that the latter is admissible on the category of  $p$ -groups, that is, compatible with the representation-theoretic process of induction.

In this paper, we show that this condition does not hold, by explicitly calculating the  $\gamma$ -filtration on the representation rings of certain extraspecial 2-groups.

### 1. BACKGROUND

The Atiyah–Hirzebruch spectral sequence provides a link between the integral cohomology ring of a finite group  $G$  and the  $K$ -theory of its

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classifying space, graded by the even, decreasing, skeletally induced filtration,

$$K_q(BG) = \ker \{ K(BG) \rightarrow K(BG^{q-1}) \}.$$

In [1], Atiyah transplanted the representation ring  $R(G)$  into this spectral sequence by means of the natural map  $\alpha: R(G) \rightarrow K(BG)$ , which becomes a topological isomorphism  $\hat{\alpha}$  upon completion with respect to the  $I$ -adic and skeletal filtrations, respectively. To replace  $K(BG)$  with  $\widehat{R(G)}$  as the abutment of the Atiyah-Hirzebruch spectral sequence requires that we define a new, even, decreasing filtration on  $R(G)$ , called the *topological filtration*, by

$$R_n^{\text{top}}(G) = \alpha^{-1}(K_n(BG)).$$

We can then create a spectral sequence

$$H^*(G; \mathbb{Z}) \Rightarrow Gr_* R^{\text{top}}(G).$$

Although the topological structure  $K(BG)$  has been replaced with the algebraic structure  $R(G)$ , the influence of topology persists in the definition of the filtration of  $R(G)$ . Atiyah sought to remove this last vestige of topology by describing an algebraically defined filtration, called the Grothendieck, or  $\gamma$ -filtration, that he conjectured would coincide with the topological filtration. This filtration is defined on any augmented  $\lambda$ -ring  $R$  as follows; for  $k \geq 0$ ,  $R_{2k}^\gamma = R_{2k-1}^\gamma$  is the additive subgroup generated by monomials of the form

$$\gamma^{i_1}(x_1) \gamma^{i_2}(x_2) \cdots \gamma^{i_n}(x_n),$$

where  $\gamma^k(x) = \lambda^k(x+k-1)$ ,  $\sum_{j=1}^n i_j \geq k$  and the  $x_j$  belong to the augmentation ideal. We will say that the element  $\gamma^i(x)$  has *weight*  $i$ ; the *total weight* of a product  $\prod_{j=1}^n \gamma^{i_j}(x_j)$  is the sum of the individual weights,  $\sum_{j=1}^n i_j$ . More generally, for any  $y \in R$ , we define the *maximum weight* of  $y$  to be the largest integer  $k$  such that  $y \in R_{2k}^\gamma$ .

Exterior powers can be used to give  $R(G)$  the structure of an augmented  $\lambda$ -ring; the resulting  $\gamma$ -filtration is finer than the topological filtration. Atiyah conjectured that in fact  $R_{2k}^\gamma(G) = R_{2k}^{\text{top}}(G)$ , for all finite groups  $G$  and for all  $k \geq 0$ . Specifically, he proved that the conjecture holds for any group  $G$  when  $k \in \{0, 1, 2\}$ , and for all  $k$  if  $G$  is either the symmetric group  $S_3$ , the group of quaternions  $Q$ , or an elementary abelian  $p$ -group. In addition, in the appendix to his paper, Atiyah outlined an intriguing connection between his conjecture and the Chern subring of the group cohomology. If  $\rho \in R(G)$ , then to the vector bundle  $\xi_\rho$  we may associate Chern classes  $c_i(\rho) \in H^{2i}(G, \mathbb{Z})$ , which generate a subring of  $H^{\text{even}}(G, \mathbb{Z})$  denoted by

$\text{Ch}(G)$ . Then  $\text{Ch}(G)$  is contained in the subgroup of universal cycles in the spectral sequence

$$H^*(G, \mathbb{Z}) \Rightarrow Gr_* R^{top}(G),$$

and moreover,  $R_{2k}^\gamma(G) = R_{2k}^{top}(G)$  for all  $k \geq 0$  if and only if the Chern subring  $\text{Ch}(G)$  is mapped surjectively onto  $Gr_* R^{top}(G)$  by the natural epimorphism. Thus, in particular, the two filtrations agree on all finite groups for which the Chern ring exhausts the even-dimensional cohomology.

However, counter-examples to Atiyah's conjecture have been discovered. In his doctoral thesis of 1972, A. C. Weiss showed by direct computation that the alternating group on four letters has the property that  $R_{4k+2}^\gamma(A_4)$  is contained as a subgroup of index 2 in  $R_{4k+2}^{top}(A_4)$ . In 1983, C. B. Thomas generalized this to a whole family of counter-examples,  $PSL_2(\mathbb{F}_l)$ , where  $l \equiv \pm 3 \pmod{8}$  [11]. I. Leary and N. Yagita have recently found a counter-example in the category of  $p$ -groups [6], namely

$$\langle A, B, C: A^p = B^p = C^{p^{s+1}} = [B, C] = 1, [A, C] = B, [A, B] = C^{sp^s} \rangle,$$

where  $p \geq 5$  and  $s$  is either 1 or a quadratic non-residue modulo  $p$ .

## 2. ADMISSIBLE FILTRATIONS

Although Atiyah's conjecture has proved false, there is still reason to hope that the topological filtration is somehow algebraically determined. To this end, J. Ritter introduced in his thesis of 1970 the concept of admissible filtrations [9].

**DEFINITION.** An even, decreasing filtration  $\{R_{2k}^a(G)\}$  on the representation ring of a finite group  $G$  is called *admissible* if:

- (1) (functoriality) for any group homomorphism  $h: G \rightarrow \Gamma$ ,  $h^*(R_{2k}^a(\Gamma)) \subseteq R_{2k}^a(G)$ ;
- (2)  $R_0^a(G) = R(G)$  and  $R_1^a(G) = I(G)$ ;
- (3) for any  $k, l$ ,  $R_{2k}^a(G) \cdot R_{2l}^a(G) \subseteq R_{2(k+l)}^a(G)$ ;
- (4) if  $G = C$  is cyclic, then  $R_{2k}^a(G) = I(G)^k$ ;
- (5)  $R_2^a(G)/R_4^a(G) \cong H^2(G, \mathbb{Z})$ ;
- (6) for all subgroups  $H \leq G$  and all  $k \geq 0$ ,  $Ind_H^G(R_{2k}^a(H)) \subseteq R_{2k}^a(G)$ .

We shall refer to (6) as the *induction condition*.

The  $I$ -adic filtration is admissible, for example, and in fact it is the unique admissible filtration on the category of abelian groups. Any

admissible filtration on the category of  $p$ -groups extends uniquely to an admissible filtration on the category of all finite groups via

$$R_{2k}^a(G) \equiv \bigcap_{P \leq G} \text{Ind}_P^G(R_{2k}^a(P)), \quad (*)$$

where the intersection runs over all  $p$ -subgroups  $P$  of  $G$ .

It is not difficult to show that the topological filtration is admissible on the category of finite groups, and that the  $\gamma$ -filtration satisfies conditions (1) through (5). However, C. B. Thomas' family of counter-examples,  $PSL_2(\mathbb{F}_l)$ ,  $l \equiv \pm 3 \pmod{8}$ , shows that the  $\gamma$ -filtration does not in general satisfy the induction condition, because it is known that topological and  $\gamma$ -filtrations coincide on all  $p$ -subgroups of such groups. On the positive side, it is evident that the topological filtration on  $PSL_2(\mathbb{F}_l)$ ,  $l \equiv \pm 3 \pmod{8}$  is nonetheless algebraically determined, by the  $\gamma$ -filtration on its  $p$ -subgroups and by (\*) above. It is therefore reasonable to pose the question, as C. B. Thomas did in [11], of whether the  $\gamma$ -filtration is admissible on the category of  $p$ -groups.

It is important to note that Leary and Nagita's counter-example does not necessarily answer this question in the negative. It is possible for a group to have more than one admissible filtration, as Ritter showed for the group of quaternions [9]. However, knowing that the  $\gamma$ -filtration is admissible on the category of  $p$ -groups would enable us to characterize the topological filtration algebraically on this category, at least up to the inversion of certain elements. More precisely, suppose  $x \in R_{2k}^{\text{top}}(P)$  and  $H$  is a subgroup of  $P$  on which the two filtrations agree. Then by functoriality,

$$\text{Res}_H^P(x) \in R_{2k}^{\text{top}}(H) = R_{2k}^\gamma(H).$$

By reciprocity, we can conclude

$$\text{Ind}_H^P \circ \text{Res}_H^P(x) = x \cdot \text{Ind}_H^P(1) \in R_{2k}^\gamma(P).$$

In order to pursue C. B. Thomas' question, it makes sense to study the  $\gamma$ -filtration on  $p$ -groups  $P$  for which it is known that the two filtrations do not coincide; this is guaranteed to be the case if  $\text{Ch}(P) \not\subseteq H^{\text{even}}(P, \mathbb{Z})$ . It has been shown that this condition is satisfied by the family of extra-special  $p$ -groups (see [4] for  $p=2$ , [10] for more general primes).

*Remark.* The result above for  $p=2$  is surprising in light of Quillen's work in [7], in which he shows that the mod-2 cohomology of extra-special 2-groups is generated by the Stiefel-Whitney classes (which are the "mod-2 equivalent" of Chern classes) associated to complex representations of the group.

3. THE EXTRA-SPECIAL  $p$ -GROUPS  $D^r$ 

In the following,  $P'$ ,  $\Phi(P)$ , and  $Z(P)$  denote the commutator subgroup, the Frattini subgroup, and the center of  $P$ , respectively.

DEFINITION. A  $p$ -group  $P$  is called *special* if either

- (1)  $P$  is elementary abelian, or
- (2)  $P$  is of class 2 and  $P' = \Phi(P) = Z(P)$  is elementary abelian.

A special  $p$ -group is called *extra-special* if  $P' = \Phi(P) = Z(P) \cong \mathbb{Z}/p$ .

The structure of extra-special  $p$ -groups is well known; see, for example, [3]. In this paper, we consider the extra-special 2-groups that are central products of  $r$  copies of the dihedral group; such groups have presentation

$$D^r = \langle A_i, B_i, C, 1 \leq i \leq r : A_i^2 = B_i^2 = C^2 = 1; \\ [A_i, B_i] = C \text{ is the only non-trivial commutator} \rangle.$$

Note that  $Z = \{1, C\}$ ; also,  $D^r$  has order  $2^{2r+1}$  and exponent 4. The number of elements of order 4 and non-central elements of order 2 are counted in the proposition below.

PROPOSITION 3.1. In  $D^r$ , we define the subsets

$$F_r = \{y \in D^r : |y| = 4\} \quad \text{and} \quad T_r = \{y \in D^r - Z, : |y| = 2\}.$$

Then  $|F_r| = 2^{2r} - 2^r$  and  $|T_r| = 2^{2r} + 2^r - 2$ .

*Proof.* Elements of  $D^r$  can be written uniquely in the form

$$y = A_1^{e_1} \cdots A_r^{e_r} B_1^{f_1} \cdots B_r^{f_r} C^\varepsilon,$$

where  $e_i, f_j, \varepsilon \in \{0, 1\}$ . If  $e_i = 1$ , then we will say  $y$  contains  $A_i$ ; similarly for  $f_j$  and  $B_j$ , and  $\varepsilon$  and  $C$ , respectively;  $y$  is of order 4 if and only if it contains an odd number of pairs  $\{A_i, B_i\}$ .

Suppose  $y$  contains exactly  $k$  pairs  $\{A_i, B_i\}$ , where  $k$  is an odd integer less than  $r$ . There are  $\binom{r}{k}$  choices of these  $k$  indices; for the remaining  $r - k$  indices,  $y$  contains either  $A_j$  or  $B_j$  or neither, but not both, and  $y$  may or may not contain  $C$ . This accounts for  $\binom{r}{k} \cdot 3^{r-k} \cdot 2$  such elements of order 4. Thus, in total we have

$$|F_r| = 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^r \binom{r}{k} 3^{r-k}.$$

This sum can be evaluated using the generating function  $f(X) = (X+3)^r$ . Then  $|F_r| = f(1) - f(-1) = 4^r - 2^r = 2^{2r} - 2^r$ . Finally,  $|T_r| = |D^r| - |F_r| - 2$ . ■

It will be important in what follows to establish that the automorphism group of  $D^r$  operates transitively on  $F_r$  and  $T_r$ ; for this we turn to the theory of bilinear forms. In the central extension of any extra-special 2-group  $P$  of order  $2^{2r+1}$

$$0 \longrightarrow Z(P) \longrightarrow P \xrightarrow{\pi} V \longrightarrow 0,$$

$V$  is an elementary abelian 2-group, and may be regarded as a  $2r$ -dimensional vector space over  $Z(P) \cong \mathbb{Z}/2$ . Define a bilinear form on  $V$  by  $B(w, y) = [\tilde{w}, \tilde{y}]$ , where  $\tilde{w}$  and  $\tilde{y}$  are pre-images of  $w$  and  $y$  under  $\pi$ , respectively;  $(V, B)$  is a symplectic space. Let  $\{w_1, y_1, \dots, w_r, y_r\}$  denote a hyperbolic basis for  $V$  over  $\mathbb{Z}/2$ . Associated to  $B$  is the quadratic form  $Q(v) = \tilde{v}^2$  making  $(V, Q)$  an orthogonal space. Now, any automorphism  $\Omega \in \text{Aut}(P)$  acts as the identity on  $Z(P)$ , and therefore induces a linear automorphism  $\bar{\Omega}$  on  $V$  which is in fact an element of the group of isometries  $O(V, Q)$ .

**PROPOSITION 3.2.** *Let  $P$  be an extra-special 2-group, with unique non-trivial central element  $C$ . Let  $\text{Inn}(P)$  denote the group of inner automorphisms of  $P$ . Then the sequence below is exact:*

$$1 \rightarrow \text{Inn}(P) \rightarrow \text{Aut}(P) \rightarrow O(V, Q) \rightarrow 1.$$

A proof may be found in [5].

**PROPOSITION 3.3.** *Let  $T = \{x \in P - Z(P) : x^2 = 1\}$  and  $F = \{x \in P - T : x^4 = 1\}$ . Then  $\text{Aut}(P)$  operates transitively on  $T$  and on  $F$ .*

*Proof.* Let  $x, x'$  be distinct elements of  $T$ . If  $x = x'C$ , then some inner automorphism will bring  $x'$  to  $x$ . Hence, it suffices to show that  $O(V, Q)$  operates transitively on the set of singular elements and on the set of non-singular elements of  $V$ .

If  $r = 1$ , then  $V = \{0, w, y, w + y\}$ . The only non-singular element is  $w + y$ ; it is obvious that the two singular elements  $w$  and  $y$  may be interchanged, and so we are done.

So suppose  $r \geq 2$ , and let  $v, v' \in V$  be such that  $Q(v) = Q(v')$ . Since  $Q$  is non-degenerate, there exist  $u, u' \in V$  such that

$$B(u, v) = B(u', v') = 1 \quad \text{and} \quad Q(u) = Q(u').$$

Let  $W = \langle u, v \rangle$  and  $W^* = \langle u', v' \rangle$ ; we claim that  $(W, Q)$  and  $(W^*, Q)$  are isometric orthogonal spaces. To prove this, first note that  $W$  and  $W^*$  are

both non-singular with respect to  $Q$ ; if  $Q(u) = Q(u') = Q(v) = Q(v')$ , then  $Q(u + v) = 1 = Q(u' + v')$ . That they are isometric now follows from the fact that they are of the same dimension and have the same Arf invariant, that is,

$$Q(u) Q(v) = Q(u') Q(v').$$

By Witt's theorem, the isometry between  $W$  and  $W^*$  extends to an isometry of  $V$  into itself,  $\bar{\Omega} \in O(V, Q)$ , such that  $\bar{\Omega}(v)$  is one of  $v'$ ,  $u'$ , or  $v' + u'$ . If  $\bar{\Omega}(v) = u'$ , then we know  $Q(u') = Q(u) = Q(v) = Q(v')$ ; we can modify the isometry  $W \rightarrow W^*$  by composing it with the isometry that switches the hyperbolic basis elements of  $W^*$ . If  $\bar{\Omega}(v) = u' + v'$ , then  $Q(u) = Q(u') = 1$ ; we can modify the original isometry by composing it with  $s_{u'} : W^* \rightarrow W^*$ . Hence, we may assume  $\bar{\Omega}(v) = v'$ , and so we are done. ■

The maximal subgroups of  $P$  may be determined in a similar manner. Suppose  $M \leq P$  is maximal; by definition of the Frattini subgroup,  $\Phi(P) = Z(P) \subseteq M$ , and so  $M$  projects via  $\pi$  onto a subgroup  $W$  of  $V$ . Since  $[P : M] = 2$ ,  $W$ , as a subspace of  $V$ , has codimension 1. Thus,  $W$  has odd dimension  $2r - 1$ , and so the bilinear form  $B$  cannot be symplectic on  $W$ . Clearly, the failure must be the result of a degeneracy; that is, there is some  $w \in W$  such that  $B(w, z) = 0$  for all  $z \in W$ . Since  $B$  is non-degenerate on  $V$ , we may consider  $W$  as  $w^\perp$  within  $V$ .

Suppose  $M'$  is another maximal subgroup of  $P$ ; let  $W' = \pi(M')$  be the corresponding subspace of  $V$ . As before, there is some  $w' \in W'$  such that  $W' = (w')^\perp$  in  $V$ . If  $Q(w) = Q(w')$ , then by Proposition 3.2 there is some  $\bar{\Omega} \in O(V, Q)$  such that  $\bar{\Omega}(w) = w'$ , and therefore an automorphism  $\Omega \in \text{Aut}(P)$  such that  $\Omega(M) = M'$ . We have proved

**PROPOSITION 3.4.** *Let  $P$  be an extra-special 2-group of order  $p^{2r+1}$ , where  $r \geq 2$ . Then up to isomorphism, there are exactly two maximal subgroups of  $P$ .*

#### 4. THE $\gamma$ -FILTRATION ON $V$

Before beginning our calculations on the  $\gamma$ -filtration on the representation ring of  $D'$ , we present some general results on frequently encountered factor groups.

**PROPOSITION 4.1.** *Let  $H$  be a subgroup of a finite group  $G$  that contains  $G'$ . Let*

$$V_G = G/G' \quad \text{and} \quad V_H = H/G'.$$

We regard  $R(V_G)$  and  $R(V_H)$  as subrings of  $R(G)$  and  $R(H)$ , respectively. Then for all  $k \geq 0$ ,

$$\text{Ind}_{V_H}^{V_G}(R_{2k}^\gamma(V_H)) \subseteq R_{2k}^\gamma(V_G).$$

Thus,  $\text{Ind}_H^G(R_{2k}^\gamma(H)) \subseteq R_{2k}^\gamma(G)$  (i.e., the induction condition holds on  $H \leq G$ ) if and only if

$$\overline{\text{Ind}}(R_{2k}^\gamma(H)/R_{2k}^\gamma(V_H)) \subseteq R_{2k}^\gamma(G)/R_{2k}^\gamma(V_G),$$

where  $\overline{\text{Ind}}$  is the obvious induced map.

*Proof.*  $V_G$  and  $V_H$  are in the category of abelian groups, for which the  $\gamma$ -filtration coincides with the admissible  $I$ -adic filtration. The second statement of the proposition is an obvious consequence of the first. ■

In every situation in which we make use of Proposition 4.1,  $V_G$  and  $V_H$  will be elementary abelian. In the next proposition, we explicitly calculate the  $\gamma$ -filtration on such groups for  $p = 2$ . First recall that if  $\{a_1, a_2, \dots, a_t\}$  is a  $\mathbb{Z}$ -basis for  $I(G)$ , let  $\sigma_{i,j} = \gamma^i(a_j)$  be the  $\gamma$ -filtration generator of weight  $i$  associated to  $a_j$ . Then  $R_{2k}^\gamma(G)$  is generated over  $\mathbb{Z}$  by monomials of total weight  $\geq k$  in the generators

$$\{\sigma_{i,j} \mid j \in \{1, 2, \dots, t\}, 1 \leq i \leq \varepsilon(a_j)\}.$$

**PROPOSITION 4.2.** *Let  $V$  be an elementary abelian group of exponent 2 and rank  $n$ ; let  $\{U_1, U_2, \dots, U_n\}$  be a set of generators of  $V$  over  $\mathbb{Z}/2$ . For  $1 \leq i \leq n$ , let  $\psi_i$  denote the 1-dimensional representation given by*

$$\psi_i(U_j) = \begin{cases} -1, & \text{if } i = j \\ 1, & \text{for } i \neq j. \end{cases}$$

Set  $u_i = \psi_i - 1 \in I(V)$ . Then for all  $k \geq 1$ ,  $R_{2k}^\gamma(V)$  is generated over  $\mathbb{Z}$  by

$$\left\{ 2^m u_1^{e_1} \cdots u_n^{e_n} : e_i \in \{0, 1\}, \sum e_i \neq 0, m + \sum e_i \geq k \right\}.$$

*Proof.* We begin by showing that these monomials are contained in  $R_{2k}^\gamma(V)$  by using induction on  $k$ . For  $k = 1$ , the proposition is trivial; assume now that it holds up to  $k - 1$ . Consider the monomial

$$\mu = 2^m u_1^{e_1} \cdots u_n^{e_n}, \quad \text{where } m + \sum e_i \geq k.$$

If  $\sum e_i \geq 2$  the inductive step is easy; we can assume without loss of generality that  $e_1 = 1$  and write

$$\mu = (2^m u_1^{e_1 - 1} u_2^{e_2} \cdots u_n^{e_n}) \cdot u_1.$$



The first monomial lies in  $R_{2(k-1)}^z(V)$  by the inductive hypothesis, and  $u_1$  is an element of  $I(V) = R_{2(1)}^z(V)$ . Hence, the product lies in  $R_{2k}^z(V)$ .

The case where  $\mu = 2^m u_i$  is handled by noticing that  $1 = \psi_i^2$  implies  $u_i^2 = -2u_i$ .

To show that this set of monomials generates, recall that by definition,  $R_{2k}^z(V)$  is generated over  $\mathbb{Z}$  by

$$\left\{ u_i^{e_1} \cdots u_n^{e_n} : \sum_{i=0}^n e_i \geq k \right\}.$$

For each  $e_i \geq 2$ , we may use  $u_i^2 = -2u_i$  to break the monomial up into a linear combination of those contained in the statement of the proposition. ■

An element of  $R(V)$  that arises frequently in subsequent calculations is, in the notation of the preceding proposition,

$$s \equiv \left( \sum_{0 \leq e_i \leq p-1} \psi^{e_1} \cdots \psi^{e_n} \right) - 2^n;$$

that is,  $s$  is the augmentation ideal element corresponding to the sum of all possible 1-dimensional representations of  $V$ . If  $V$  is identified with  $P/P'$  for some 2-group  $P$ , then  $s$  can also be considered as the augmentation ideal element corresponding to the sum of all 1-dimensional representations of  $P$ .

The character of  $s$  may be computed by taking advantage of a certain symmetry in the elements and characters of  $V$ , namely,

$$\begin{aligned} \omega^{\sum e_i f_i} &= \psi_1^{e_1} \cdots \psi_n^{e_n} (U_1^{f_1} \cdots U_n^{f_n}) \\ &= \psi_1^{f_1} \cdots \psi_n^{f_n} (U_1^{e_1} \cdots U_n^{e_n}). \end{aligned}$$

This symmetry can be best expressed by adopting the notation

$$\text{for } y = U_1^{e_1} \cdots U_n^{e_n} \in V, \text{ let } \rho_y = \psi_1^{e_1} \cdots \psi_n^{e_n} \in R(V).$$

Then for all  $w, y \in V$ , we have  $\rho_y(w) = \rho_w(y)$ . The character of  $s$  evaluated at some  $y \in V$  can then be computed as

$$\begin{aligned} s(y) &= \left( \sum_{w \in V} \rho_w(y) \right) - 2^n = \left( \sum_{w \in V} \rho_y(w) \right) - 2^n \\ &= (|V| \cdot \langle 1, \rho_y \rangle) - 2^n \\ &= \begin{cases} 0, & y = 1 \\ -2^n, & y \neq 1. \end{cases} \end{aligned}$$

PROPOSITION 4.3. *The element  $s$  has maximum weight  $n$  in the  $\gamma$ -filtration on  $R(V)$ .*

*Proof.* First, we claim that multiplication by  $u_1$  on  $R_{2(e)}^s(V)$  increases weight by exactly 1. For suppose

$$y = \sum_j 2^{m_j} \mathcal{M}_j u_1^{e_{1,j}} \cdots u_n^{e_{n,j}},$$

where  $\mathcal{M}_j$  is odd. Then by Proposition 4.2, the maximum weight of  $y$  is given by

$$\min_j \left\{ m_j + \sum_{i=0}^n e_{i,j} \right\}.$$

Now, multiplying the  $j$ th summand in the expression of  $y$  by  $u_1$  results in

$$\begin{aligned} &2^{m_j} \mathcal{M}_j u_1^1 u_2^{e_{2,j}} \cdots u_n^{e_{n,j}}, & e_{1,j} = 0 \\ &-2^{m_j+1} \mathcal{M}_j u_1^q u_2^{e_{2,j}} \cdots u_n^{e_{n,j}}, & e_{1,j} = 1. \end{aligned}$$

In either case, the maximum weight of the product of  $u_1$  with the  $j$ th summand is  $m_j + \sum_i e_{i,j} + 1$ . The claim now follows.

Next, by our calculation of the character of  $s$ , we have

$$s \cdot u_1 = -2^n u_1,$$

which has maximum weight  $n + 1$ ; hence,  $s$  must have started with weight  $n$ . ■

### 5. BASIC CALCULATIONS

First we compute the representation ring  $R(D^r)$ . For  $1 \leq i \leq r$ , let  $\alpha_i$  be the 1-dimensional representation that sends  $A_i$  to  $-1$  and all other generators to 1; define  $\beta_j$  analogously. (At this point, we issue the warning that no distinction will be made between the  $\alpha_i$  and  $\beta_j$  that appear in  $R(D^r)$  for different values of  $r$ .) The set of all possible tensor products of these generators contains  $2^{2r}$  1-dimensional representations.

Notice that any sum of 1-dimensional representations may be considered a virtual character of  $V^r = D^r/Z$ . We define the  $V$ -weight of such an element to be its maximum weight in the  $\gamma$ -filtration on  $V$ .

*Remark.* If  $w \in R(V^r)$ , then  $w$  may be trivially extended to  $R(V^q)$ , for any  $q \geq r$ ; its  $V$ -weight does not change under this extension. For this reason, we will usually omit the superscript on  $V$ , unless the context demands such specification.

Let  $N_r = \langle B_1, \dots, B_r, C \rangle$ ; then  $N_r$  is a normal, abelian subgroup of  $D^r$  of index  $2^r$ . Let  $\xi_r$  be the 1-dimensional representation of  $N_r$  given by

$$\xi_r : \begin{cases} C \mapsto -1 \\ B_j \mapsto 1. \end{cases}$$

This induces a  $2^r$ -dimensional representation on  $D^r$  with character

$$\text{Ind}_{N_r}^{D^r} \xi_r(y) \equiv \text{Ind} \xi_r(y) = \begin{cases} 2^r, & y = 1 \\ -2^r, & y = C \\ 0, & y \notin Z. \end{cases}$$

It can be shown with Mackey's criterion that this is an irreducible representation of  $D^r$ .

This accounts for all the irreducible representations of  $D^r$ , as

$$|D^r| = 2^{2r+1} = 2^{2r} \cdot (1)^2 + 1 \cdot (2^r)^2.$$

Next, let  $a_i = \alpha_i - 1$  and  $b_j = \beta_j - 1$  for  $1 \leq i, j \leq r$ ; the warning issued about the  $\alpha_i$  and  $\beta_j$  also applies to the  $a_i$  and  $b_j$ . Let  $g_r = \text{Ind} \xi_r - 2^r$ . Then

$$\{g_r\} \cup \{a_1^{e_1} \cdots a_r^{e_r} b_1^{f_1} \cdots b_r^{f_r} : e_i, f_j \in \{0, 1\}, \sum e_i + \sum f_j > 0\}$$

constitutes a  $\mathbb{Z}$ -basis for the augmentation ideal,  $I(D^r)$ .

The  $\gamma$ -filtration generators of weight 1 are just the generators of the augmentation ideal described above. The only  $\gamma$ -filtration generators of weight  $> 1$  are those associated to the only representation of degree  $> 1$ ; these are of the form

$$\begin{aligned} \sigma_{k,r} &\equiv \gamma^k(g_r) = \lambda^k(\text{Ind} \xi_r - 2^r + k - 1) \\ &= \sum_{j=0}^k \lambda^j(\text{Ind} \xi_r) \cdot \lambda^{k-j}(-2^r + k - 1) \\ &= \sum_{j=0}^k (-1)^{k-j} \binom{2r-j}{k-j} \lambda^j(\text{Ind} \xi_r), \end{aligned}$$

based on the unique  $\lambda$ -ring structure on  $\mathbb{Z}$ .

The exterior powers of  $\text{Ind} \xi_r$  may be calculated using a formula described in [10]; for any representation  $\rho$  of a finite group  $G$ , the character of  $\lambda^j \rho$  is given by

$$\chi_{\lambda^j \rho}(g) = \text{sym}_j(\text{eigenvalues of } \rho(g)),$$

where  $\text{sym}_j$  denotes the  $j$ th symmetric polynomial.

The eigenvalues of  $\text{Ind}\xi_r(1)$  are clearly  $2^r$  copies of 1, and the eigenvalues of  $\text{Ind}\xi_r(C)$  are just as clearly  $2^r$  copies of  $-1$ . For  $y \in T_r$ , the eigenvalues of  $\text{Ind}\xi_r$  must be square roots of unity and, given the character of  $\text{Ind}\xi_r$ , they must add up to zero. Hence the set of eigenvalues consists of  $2^{r-1}$  copies of  $-1$  and  $2^{r-1}$  copies of 1. For  $y \in F_r$ , we have  $y^2 = C$  and so all eigenvalues are square roots of  $-1$ ; by character considerations, there must be  $2^{r-1}$  copies each of  $i$  and  $-i$  (where  $i = \sqrt{-1}$ ). The symmetric polynomials in these sets of eigenvalues appear, up to sign, as coefficients of the polynomials

$$P_1(X) = (X-1)^{2^r} = \sum_{\lambda} (-1)^{\lambda} \binom{2^r}{\lambda} X^{\lambda}$$

$$P_C(X) = (X+1)^{2^r} = \sum_{\lambda} \binom{2^r}{\lambda} X^{\lambda}$$

$$P_{T_r}(X) = (X-1)^{2^{r-1}}(X+1)^{2^{r-1}} = (X^2-1)^{2^{r-1}} = \sum_{\mu} (-1)^{\mu} \binom{2^{r-1}}{\mu} X^{2\mu}$$

$$P_{F_r}(X) = (X-i)^{2^{r-1}}(X+i)^{2^{r-1}} = (X^2+1)^{2^{r-1}} = \sum_{\mu} \binom{2^{r-1}}{\mu} X^{2\mu}.$$

The character of  $\lambda^j \text{Ind}\xi_r$  is therefore given by

$$\lambda^j \text{Ind}\xi_r(y) = \begin{cases} \binom{2^r}{j}, & y = 1 \\ (-1)^j \binom{2^r}{j}, & y = C \\ 0, & y \notin Z(D^r), j \text{ odd,} \\ (-1)^l \binom{2^{r-1}}{l}, & y \in T_r, j = 2l, \\ \binom{2^{r-1}}{l}, & y \in F_r, j = 2l. \end{cases}$$

To express  $\lambda^j \text{Ind}\xi_r$  in terms of the irreducible representations, we introduce some new notation. Let  $\overline{T}_r$  and  $\overline{F}_r$  denote the images in  $V^r$  of  $T_r$  and  $F_r$ , respectively, and set

$$\theta = \sum_{v \in \overline{T}_r} \rho_v \quad \text{and} \quad \phi = \sum_{v \in \overline{F}_r} \rho_v,$$

where  $\rho_v$  is as defined in Section 5. Notice that

$$s = s_r = (1 + \phi + \theta) - 2^r,$$

where  $s_r$  is the augmentation ideal element corresponding to the sum of all 1-dimensional representations of  $D^r$ .

**PROPOSITION 5.1.** (1) For odd  $j$ ,  $\lambda^j \text{Ind}\xi_r = (1/2^r) \binom{2^r}{j} \text{Ind}\xi_r$ .

(2) For all  $j$ ,  $\lambda^j \text{Ind}\xi_r = \lambda^{2^r-j} \text{Ind}\xi_r$ .

(3) For  $j = 2l$ ,

$$\lambda^j \text{Ind}\xi_r = M_0(l) \cdot 1 + M_1(l) \cdot \theta + M_2(l) \cdot \phi,$$

where the coefficients are given in the table

	$l$ even ( $j \equiv 0 \pmod{4}$ )	$l$ odd ( $j \equiv 2 \pmod{4}$ )
$M_0(l)$	$\frac{1}{2^{2r}} \left( \binom{2^r}{j} + (2^{2r} - 1) \binom{2^{r-1}}{l} \right)$	$\frac{1}{2^{2r}} \left( \binom{2^r}{j} - (2^r - 1) \binom{2^{r-1}}{l} \right)$
$M_1(l)$	$\frac{1}{2^{2r}} \left( \binom{2^r}{j} - \binom{2^{r-1}}{l} \right)$	$\frac{1}{2^{2r}} \left( \binom{2^r}{j} - (2^r - 1) \binom{2^{r-1}}{l} \right)$
$M_2(l)$	$\frac{1}{2^{2r}} \left( \binom{2^r}{j} - \binom{2^{r-1}}{l} \right)$	$\frac{1}{2^{2r}} \left( \binom{2^r}{j} + (2^r + 1) \binom{2^{r-1}}{l} \right)$

*Proof.* Parts (1) and (2) follow immediately from the character of  $\lambda^j \text{Ind}\xi_r$ . Assume  $j = 2l$ ; to compute  $M_0(l)$ , consider

$$\begin{aligned} \langle \lambda^j \text{Ind}\xi_r, 1 \rangle &= \frac{1}{|D^r|} \sum_{y \in D^r} (\lambda^k \text{Ind}\xi_r)(y) \\ &= \frac{1}{2^{2r+1}} \left[ 2 \binom{2^r}{j} + \sum_{y \in T_r} (-1)^l \binom{2^{r-1}}{l} + \sum_{y \in F_r} \binom{2^{r-1}}{l} \right] \\ &= \frac{1}{2^{2r+1}} \left[ 2 \binom{2^r}{j} + ((-1)^l |T_r| + |F_r|) \binom{2^{r-1}}{l} \right]. \end{aligned}$$

Now use Proposition 3.1. To compute the coefficients  $M_1(l)$  and  $M_2(l)$ , we must first find the character of  $\theta$  and  $\phi$ .

**LEMMA 5.2.**

$$\theta(y) = \begin{cases} 2^{2r-1} + 2^{r-1} - 1, & y \in Z, \\ 2^{r-1} - 1, & y \in T_r, \\ -2^{r-1} - 1, & y \in F_r. \end{cases}$$

$$\phi(y) = \begin{cases} 2^{2r-1} - 2^{r-1}, & y \in Z, \\ -2^{r-1}, & y \in T_r, \\ 2^{r-1}, & y \in F_r. \end{cases}$$

*Proof.* For any  $y \in D'$ ,

$$\begin{aligned}\theta(y) &= \sum_{v \in T_r} \rho_v(y) = \sum_{v \in T_r} \rho_{\bar{y}}(v), \quad \text{where } \bar{y} = \pi(y) \in V' \\ &= \frac{1}{2} \sum_{\bar{v} \in T_r} \rho_{\bar{y}}(\bar{v}).\end{aligned}$$

For  $y \in Z$ ,  $\rho_{\bar{y}} = 1$ , and so the first part of the character of  $\theta$  is clear.

We now show that  $\theta$  is constant on elements of  $T_r$ . Let  $A$  be the automorphism of  $D'$  defined by

$$A(A_i) = B_i, \quad A(B_i) = A_i \quad \text{for } 1 \leq i \leq r.$$

Then for all  $w, y \in D'$ , it can be checked that

$$\rho_w(y) = [A(w), y] = \begin{cases} 1, & \text{if } A(w) \text{ and } y \text{ commute,} \\ -1, & \text{otherwise.} \end{cases}$$

For any  $y \in T_r$ ,

$$\theta(y) = \frac{1}{2} \sum_{w \in T_r} \rho_w(y) = \frac{1}{2} \sum_{w \in T_r} [A(w), y] = \frac{1}{2} \sum_{w \in T_r} [w, y],$$

because  $T_r$  is a characteristic subset of  $D'$ . Now suppose  $y'$  is another element of  $T_r$ ; then by Proposition 3.3, we know that there exists  $A \in \text{Aut}(D')$  such that  $A(y) = y'$ . Consider

$$\begin{aligned}\theta(y') &= \frac{1}{2} \sum_{w \in T_r} [w, y'] = \frac{1}{2} \sum_{w \in T_r} [w, Ay] \\ &= \frac{1}{2} \sum_{w \in T_r} [Aw, Ay] = \frac{1}{2} \sum_{w \in T_r} [w, y] = \theta(y),\end{aligned}$$

since  $A$  acts as the identity on  $(D')' = Z$ . A similar argument works to show that  $\theta$  is constant on elements of  $F_r$ , and to show that  $\phi$  is constant on elements of  $T_r$  and  $F_r$ .

To compute the character of  $\theta$ , therefore, we need only evaluate it on judiciously chosen elements of  $T_r$  and  $F_r$ . In particular, we have

$$\begin{aligned}\theta(A_r) &= \frac{1}{2} \sum_{w \in T_r} [w, A_r] \\ &= \frac{1}{2} (|\{w \in T_r : w \text{ does not contain } B_r\}| - |\{w \in T_r : w \text{ contains } B_r\}|).\end{aligned}$$

Now,

$$\{w \in T_r : w \text{ does not contain } B_r\} = \{w', w'A_r : w' \in T_{r-1}\} \cup \{A_r, A_r C\}$$

and

$$\begin{aligned} & \{w \in T_r : w \text{ contains } B_r\} \\ &= |\{w' B_r : w' \in T_{r-1}\} \cup \{w' A_r B_r : w' \in F_{r-1}\} \cup \{B_r, B_r C\}. \end{aligned}$$

Applying Proposition 3.1 yields

$$\theta(A_r) = \frac{1}{2} ((2^{2r-1} + 2^r - 2) - 2^{2r-1}) = 2^{r-1} - 1.$$

The remaining results in the lemma may be computed more indirectly. Let  $\zeta$  represent the character of  $\theta$  on elements of  $F_r$ . Then

$$\begin{aligned} 0 &= |D^r| \langle \theta, 1 \rangle = \sum_{y \in D^r} \theta(y) \\ &= 2(2^{2r-1} + 2^{r-1} - 1) + |T_r| (2^{r-1} - 1) + |F_r| \zeta. \end{aligned}$$

We can solve for  $\zeta$  with the aid of Proposition 3.1 to get  $\zeta = -2^{r-1} - 1$ . Finally, the character of  $\phi$  is given by  $s = (1 + \theta + \phi) - 2^r$ ;

$$\begin{aligned} \phi(A_r) &= (s - 1 - \theta + 2^r)(A_r) = -2^{r-1}; \\ \phi(A_r B_r) &= (s - 1 - \theta + 2^r)(A_r B_r) = 2^{r-1}. \quad \blacksquare \end{aligned}$$

We now return to the proof of Proposition 5.1, in which we are considering the case  $j = 2l$ . If  $l$  is even, then the character of  $\lambda^j \text{Ind} \xi_r$  may be simplified to

$$\lambda^j \text{Ind} \xi_r = \begin{cases} \binom{2^r}{j}, & y \in Z, \\ \binom{2^{r-1}}{l}, & y \notin Z. \end{cases}$$

Since

$$(\theta + \phi)(y) = \begin{cases} 2^{2r-1}, & y \in Z, \\ -1 & y \notin Z, \end{cases}$$

we may conclude that in the expression for  $\lambda^j \text{Ind} \xi_r$ ,

$$M_1(l) = M_2(l) = - \left[ \binom{2^{r-1}}{l} - M_0(l) \right] = \frac{1}{2^{2r}} \left[ \binom{2^r}{j} - \binom{2^{r-1}}{l} \right].$$

Now suppose  $j = 2l$  where  $l$  is odd. Since  $\lambda^j \text{Ind} \xi_r$  is constant on  $Z$ ,  $T_r$ , and  $F_r$ , it is clear that it can be expressed as a linear combination of  $1, \theta,$

and  $\phi$ . This means that if  $y, y'$  are non-central elements of the same order, then

$$\langle \lambda^j \text{Ind}_{\xi_r}, \rho_{\bar{y}} \rangle = \langle \lambda^j \text{Ind}_{\xi_r}, \rho_{\bar{y}'} \rangle,$$

and so we may calculate the remaining coefficients by

$$M_1(l) = \frac{1}{|T_r|} \langle \lambda^j \text{Ind}_{\xi_r}, \theta \rangle,$$

$$M_2(l) = \frac{1}{|F_r|} \langle \lambda^j \text{Ind}_{\xi_r}, \theta \rangle. \blacksquare$$

The higher weight generators of the  $\gamma$ -filtration on  $R(D')$  are therefore of the form

$$\begin{aligned} \sigma_{k,r} = & \sum_{\substack{j=1 \\ j \text{ odd}}}^k (-1)^{k+1} \binom{2^r-j}{k-j} \cdot \frac{1}{2^r} \binom{2^r}{j} \text{Ind}_{\xi_r} \\ & + \sum_{\substack{j=0 \\ j=2l}}^k (-1)^k \binom{2^r-j}{k-j} [M_0(l) \cdot 1 + M_1(l) \cdot \theta + M_2(l) \cdot \phi]. \end{aligned}$$

Since we know that  $\sigma_{k,r}$  lies in the augmentation ideal, we can replace  $\text{Ind}_{\xi_r}$  with  $g_r$ ,  $\theta$  with  $t_r \equiv \theta - |T_r|$ , and  $\phi$  with  $f_r \equiv \phi - |F_r|$ , and assume that all copies of the trivial representation get cancelled out.

The coefficient of  $g_r$  can be rewritten using an elementary combinatorial identity [8]:

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^k \binom{2^r-j}{k-j} \binom{2^r}{j} = \sum_{\substack{j=1 \\ j \text{ odd}}}^k \binom{2^r}{k} \binom{k}{j} = \binom{2^r}{k} \sum_{\substack{j=1 \\ j \text{ odd}}}^k \binom{k}{j} = \binom{2^r}{k} 2^{k-1}.$$

To simplify the expressions of the remaining coefficients, let

$$\wp_{\text{even}}(k, r) = \frac{1}{2^{2r}} \sum_{\substack{l=0 \\ l \text{ even}}}^{\lfloor k/2 \rfloor} \binom{2^r-2l}{k-2l} \binom{2^r-1}{l};$$

let  $\wp_{\text{odd}}(k, r)$  denote the analogous sum over all odd values of the index  $l$ .

LEMMA 5.3.

$$\sigma_{k,r} = L_0(k, r) g_r + L_1(k, r) t_r + L_2(k, r) f_r,$$



where

$$L_0(k, r) = (-1)^{k+1} \binom{2^r}{k} 2^{k-r-1},$$

$$L_1(k, r) = (-1)^k \left[ \binom{2^r}{k} 2^{k-2r-1} + (\wp_{\text{odd}}(k, r) - \wp_{\text{even}}(k, r)) \right. \\ \left. - 2^r \wp_{\text{odd}}(k, r) \right],$$

$$L_2(k, r) = L_1(k, r) + (-1)^k 2^{r+1} \wp_{\text{odd}}(k, r).$$

In the following corollary, we present a few special cases of these new coefficients.

COROLLARY 5.4. (1) For any  $r$ ,  $\sigma_{2^r, r} = f_r - (2^r - 1)g_r$ .

(2) The top-weight generator is

$$\sigma_{2^r, r} = -2^{2^r-r-1}g_r + 2^{2^r-1-r-1}[(2^{2^r-1-r} - 1)t_r + (2^{2^r-1-r} + 1)f_r].$$

(3) The generator of penultimate weight is redundant, as

$$\sigma_{2^r-1, r} = -2^{r-1}\sigma_{2^r, r}.$$

*Proof.* The computations are straightforward; in cases (2) and (3), notice that

$$\wp_{\text{even}}(k, r) = \wp_{\text{odd}}(k, r). \blacksquare$$

### 6. A CHANGE OF BASIS

We have already mentioned the fact that

$$s_r = (1 + \theta + \phi) - 2^{2^r} = t_r + f_r.$$

Since  $s_r$  has several nice properties that we would like to exploit, it is advantageous to switch from using  $\{t_r, f_r\}$  to using  $\{t_r + f_r, f_r - t_r\}$  in our description of the  $\gamma$ -filtration on  $R(D^r)$ . We let  $d_r$  denote the difference  $f_r - t_r$ ; from Lemma 5.2, the character of  $d_r$  is

$$d_r(y) = \begin{cases} 0, & y \in Z \text{ or } T_r, \\ -2^{r+1}, & y \in F_r. \end{cases}$$

(Hence,  $d_r$  acts as a sort of characteristic function on the the elements of order 4.) The higher weight  $\gamma$ -filtration generators can be written as

$$\sigma_{k, r} = K_0(k, r) \cdot g_r + K_1(k, r) \cdot s_r + K_2(k, r) \cdot d_r,$$

where  $K_0 = L_0$ ,  $K_1 = \frac{1}{2}(L_1 + L_2)$ , and  $K_2 = \frac{1}{2}(L_2 - L_1)$ . We know that  $L_1$  and  $L_2$  are integers, but we are only guaranteed that  $K_1, K_2 \in \mathbb{Z}[1/2]$ .

LEMMA 6.1. For any  $r$ ,

- (1)  $\sigma_{2^r, r} = -(2^r - 1)g_r + \frac{1}{2}s_r + \frac{1}{2}d_r$ ,
- (2)  $\sigma_{2^r, r} = -2^{2^r - r - 1}g_r + 2^{2^r - 2r - 1}s_r + 2^{2^r - 1 - r}d_r$ .

*Proof.* This is just a rewriting of the results of Corollary 5.4. ■

LEMMA 6.2. More generally we have

$$K_1(k, r) = (-1)^k 2^{k-2r-1} \left[ \binom{2^r}{k} - 2 \binom{2^r-1}{k} \right].$$

*Proof.* By referring to Lemma 5.3, we have

$$\begin{aligned} K_1(k, r) &= (-1)^k \left[ \binom{2^r}{k} 2^{k-2r-1} + (\wp_{\text{odd}}(k, r) - \wp_{\text{even}}(k, r)) \right] \\ &= (-1)^k \left[ \binom{2^r}{k} 2^{k-2r-1} - \frac{1}{2^{2^r}} \sum_{l=0}^{\lfloor k/2 \rfloor} (-1)^l \binom{2^r-2l}{k-2l} \binom{2^r-1}{l} \right]. \end{aligned}$$

The sum in this expression can be simplified by considering the generating function

$$\begin{aligned} P_m(X) &= [-X^2 + (X+1)^2]^m \\ &= \sum_{l=0}^m (-1)^l \binom{m}{l} (X+1)^{2m-2l} \\ &= \sum_{l=0}^m \sum_{\lambda=0}^{2m-2l} (-1)^l \binom{m}{l} \binom{2m-2l}{\lambda} X^{2l+\lambda} \\ &= \sum_{\lambda=0}^{2m} \left[ \sum_{l=0}^{2m} (-1)^l \binom{m}{l} \binom{2m-2l}{\lambda} \right] X^{2l+\lambda}. \end{aligned}$$

If we set  $m = 2^{r-1}$ , then the coefficient of  $X^k$  in the polynomial above is the sum that appears in the expression for  $K_1(k, r)$ . This coefficient can also be calculated in the following manner:

$$\begin{aligned} P_m(X) &= [-X^2 + X^2 + 2X + 1]^m = [2X + 1]^m \\ &= \sum_{\mu=0}^m \binom{m}{\mu} 2^\mu X^\mu. \end{aligned}$$

Now equate the  $k$ th coefficients for  $m = 2^{r-1}$ . ■

COROLLARY 6.3. *If  $K_1(k, r) \in \mathbb{Z}$ , then for  $k < 2^r$ ,*

$$\sigma_{k,r} \equiv K_0(k, r) g_r + K_2(k, r) d_r \pmod{R_{2k}^2(V)}.$$

*Proof.* By Proposition 4.3,  $s$ , has  $V$ -weight  $2r$ , and so the result is trivial for  $k \leq 2r$ . For  $k > 2r$ , by Proposition 4.2, it suffices to show that the 2-adic absolute value of  $K_1(k, r)$ , denoted by  $v_2[K_1(k, r)]$ , exceeds  $k - 2r$ . By Lemma 6.2, we need only show

$$v_2 \left[ \binom{2^r}{k} - 2 \binom{2^{r-1}}{k} \right] \geq 1.$$

SUBLEMMA 6.4. *Let  $p$  be a prime. Then for any  $k \leq p^r$ ,*

$$v_p \left[ \binom{p^r}{k} \right] = r - v_p(k).$$

*Proof.* In [10], C. B. Thomas proves that if  $k$  is written in  $p$ -adic expansion,

$$k = (a_0 + a_1 p + \dots + a_s p^s) p^t,$$

where  $0 \leq a_j \leq p - 1$ ,  $a_0 \neq 0$ , then

$$v_p(k!) = \frac{k - (a_0 + a_1 + \dots + a_s)}{p - 1}.$$

The sublemma follows directly from this result. ■

Returning to the proof of Corollary 6.3, first notice that if  $k > 2^{r-1}$ , then  $\binom{2^{r-1}}{k} = 0$ , and so

$$v_2 \left[ \binom{2^r}{k} - 2 \binom{2^{r-1}}{k} \right] = r - v_2(k) \geq 1,$$

since we are assuming that  $k < 2^r$ . If  $k \leq 2^{r-1}$ , then

$$v_2 \left[ \binom{2^r}{k} \right] = v_2 \left[ 2 \binom{2^{r-1}}{k} \right] = r - v_2(k);$$

hence, the 2-adic absolute value of the difference is  $\geq r - v_2(k) + 1 \geq 1$ . ■

LEMMA 6.5. (1)  $f_r$  and  $t_r$  each have  $V$ -weight  $r$ ;

(2)  $s_r$  has  $V$ -weight  $2r$ ;

(3)  $d_r$  has  $V$ -weight  $r + 1$ .

*Proof.* We have already calculated the  $V$ -weight of  $s_r$  in Proposition 4.3. The  $V$ -weight of  $d_r$  may be calculated in a similar way; from the character of  $d_r$  and Proposition 4.2, we know that multiplication by  $d_r$  in  $R(V)$  increases  $\gamma$ -filtration weight by  $r+1$ . A more direct proof involves induction on  $r$ .

For (1), use the fact that the order of an element of  $D^r$  depends on the parity of the number of pairs  $\{A_i, B_i\}$  it contains to establish the recursion formulas

$$\begin{aligned}\phi_{r+1} &= \phi_r(1 + \alpha_{r+1} + \beta_{r+1}) + \theta_r(\alpha_{r+1}\beta_{r+1}) + \alpha_{r+1}\beta_{r+1}, \\ \theta_{r+1} &= \theta_r(1 + \alpha_{r+1} + \beta_{r+1}) + \phi_r(\alpha_{r+1}\beta_{r+1}) + \alpha_{r+1} + \beta_{r+1}.\end{aligned}$$

Rewriting in terms of augmentation ideal elements gives

$$\begin{aligned}f_{r+1} &= s_r(1 + a_{r+1} + b_{r+1}) + 2f_r + a_{r+1}b_{r+1}[t_r + 2^{r-1}(2^r + 1)] \\ &\quad + 2^{2r}(a_{r+1} + b_{r+1}), \\ t_{r+1} &= s_r(1 + a_{r+1} + b_{r+1}) + 2t_r + a_{r+1}b_{r+1}[f_r + 2^{r-1}(2^r - 1)] \\ &\quad + 2^{2r}(a_{r+1} + b_{r+1}).\end{aligned}$$

To start the induction, we can directly compute

$$f_1 = \alpha_1\beta_1 - 1 = a_1b_1 + a_1 + b_1 \quad \text{and} \quad t_1 = a_1 + b_1;$$

both of these clearly have  $V$ -weight 1. The inductive step is now an easy consequence of Proposition 4.3. For the direct proof of the  $V$ -weight of  $d_r$ , notice that  $d_1 = a_1b_1$  and that

$$d_{r+1} = d_r(2 - a_{r+1}b_{r+1}) + 2^r a_{r+1}b_{r+1}. \quad \blacksquare$$

*Remark.* The previous line shows that

$$d_r \equiv 2^{r-1} \sum_{i=1}^r a_i b_i \pmod{R_{2(r+2)}^\gamma(V)}.$$

Recall that the  $k$ th  $\gamma$ -filtration ideal on  $D^r$  is generated over  $\mathbb{Z}$  by monomials of the form

$$\sigma_{i_1, r} \cdot \sigma_{i_2, r} \cdots \sigma_{i_n, r} \cdot v,$$

where  $v \in R_{2(i_{n+1})}^\gamma(V)$  ( $v$  is possibly trivial) and  $\sum_{j=1}^{n+1} i_j \geq k$ . To express such monomials in terms of our chosen basis for  $I(D^r)$ , we must investigate the product structure among the generators. Examining characters yields the list

$$\begin{aligned}g_r v &= -2^r v, & s_r v &= -2^{2r} v, & \text{for any } v \in V; \\ d_r^2 &= -2^{r+1} d_r, & g_r^2 &= s_r - 2^{r+1} g_r.\end{aligned} \tag{6.6}$$

The next lemma greatly simplifies the expression of  $\gamma$ -filtration generators modulo the  $\gamma$ -filtration on  $R(V)$ .

LEMMA 6.7. For  $v \in R_{2k}^\gamma(V)$  and  $2 \leq i \leq 2^r$ ,

$$\sigma_{i,r} v \equiv K_2(i,r) d_r v \pmod{R_{2(k+i)}^\gamma(V)}.$$

Proof. By (6.6), we have

$$\sigma_{i,r} v = K_0(i,r)(-2^r v) + K_1(i,r)(-2^{2r} v) + K_2(i,r) d_r v.$$

Now,

$$\begin{aligned} K_0(i,r)(-2^r v) &= (-1)^{i+1} (-2^r) \binom{2^r}{i} 2^{i-r-1} v = \pm 2^{i-1} \binom{2^r}{i} v \\ &= \pm 2^{i-1+r-v_2(i)} \mathcal{M} \cdot v, \end{aligned}$$

for some odd integer  $\mathcal{M}$  (using Sublemma 6.4). By Proposition 4.3, this element has  $V$ -weight  $(i-1+r-v_2(i)+k)$ . The restrictions on  $i$  in the hypothesis of the lemma guarantee that  $r-v_2(i)-1 \geq 0$ , and so  $K_0(i,r)(-2^r v) \in R_{2(k+i)}^\gamma(V)$ .

Next, if  $K_1(i,r) \in \mathbb{Z}$ , then we are done. If  $K_1(i,r) \in \mathbb{Z}[1/2]$ , we have

$$v_2[K_1(i,r)] = i-r-v_2(i) = -1.$$

But then

$$2r + v_2[K_1(i,r)] = i + (r - v_2(i)) \geq i + 1,$$

as  $i < 2^r$ . Hence  $K_1(i,r)(-2^{2r} v) \in R_{2(k+i)}^\gamma(V)$ . ■

### 7. EXAMPLES

In [10], C. B. Thomas used the Hochschild–Serre spectral sequence of the central extension to show that  $\text{Ch}(D^1) = H^{\text{even}}(D^1, \mathbb{Z})$ . Thus the topological and  $\gamma$ -filtrations coincide, meaning in particular that the  $\gamma$ -filtration on  $R(D^1)$  is admissible.

It turns out that the  $\gamma$ -filtration on  $R(D^2)$  is also admissible. As it is not known whether the Chern subring of  $D^2$  exhausts the even-dimensional cohomology, we resort to more direct methods. Many of the calculations become very long and involved, and so only a brief outline is provided here; full details can be found in [2].

In general, to show that the  $\gamma$ -filtration is admissible on a given group  $G$ , we must show that the induction condition holds for all subgroups  $H \leq G$ . By the transitivity of induction, this can be accomplished by

showing that the induction condition holds for all maximal subgroups, and that the maximal subgroups themselves have admissible  $\gamma$ -filtration. As noted in Section 3, up to isomorphism  $D^2$  has exactly two maximal subgroups,  $H \equiv D^1 \times \langle A_2 \rangle$  and  $J \equiv \langle D^1, A_2 B_2 \rangle$ .

Because  $H$  is a direct product,

$$R(H) = R(D^1) \otimes R(\langle A_1 \rangle).$$

Let  $\Psi_{D^1}: R(D^1) \rightarrow R(H)$  be the ring homomorphism given on the representation level by  $\rho \mapsto \rho \otimes 1$ ; let  $\Psi_{\langle A_2 \rangle}: R(\langle A_2 \rangle) \rightarrow R(H)$  be given by  $\alpha_2 \mapsto 1 \otimes \alpha_2$ . It is not difficult to show that  $\Psi_{D^1}$  and  $\Psi_{\langle A_2 \rangle}$  are  $\lambda$ -ring homomorphisms, and therefore preserve  $\gamma$ -filtrations. In fact, we have

$$R_{2k}^\gamma(H) = \sum_{j=0}^k \Psi_{D^1}(R_{2(j)}^\gamma(D^1)) \cdot \Psi_{\langle A_2 \rangle}(R_{2(k-j)}^\gamma(\langle A_2 \rangle)).$$

The sum on the right is clearly contained in  $R_{2k}^\gamma(H)$ ; the opposite inclusion follows from the fact that each  $\gamma$ -filtration generator of  $R(H)$  may be shown directly to lie in the given sum of ideals.

*Remark.* It should be noted at this point that this result need not generalize to arbitrary direct products of groups. In [9], Ritter showed that if  $R_{2k}^a(-)$  is an admissible filtration on a category of finite groups containing  $G$ ,  $\Gamma$ , and  $G \times \Gamma$ , then the filtration on  $G \times \Gamma$  given by

$$R_{2k}^a(G \times \Gamma) \equiv \sum_{j=0}^k \Psi_G(R_{2j}^a(G)) \cdot \Psi_\Gamma(R_{2(k-j)}^a(\Gamma))$$

is admissible, but it need not coincide with  $R_{2k}^a(G \times \Gamma)$ .

To show that the  $\gamma$ -filtration on  $H$  is admissible, first note that the maximal subgroups of  $H$  are of two types,  $D^1$  and  $L \times \langle A_2 \rangle$ , where  $L$  is a maximal subgroup of  $D^1$ . We noted that  $D^1$  has admissible  $\gamma$ -filtration; the same is true of  $L \times \langle A_2 \rangle$  because it is abelian. The induction condition on  $D^1 < H$  is covered by the following lemma.

**LEMMA 7.1.** *Let  $\Gamma < G$ . If  $\text{Res}_\Gamma^G: R(G) \rightarrow R(\Gamma)$  is surjective, then  $\text{Ind}_\Gamma^G(R_{2k}^\gamma(\Gamma)) \subseteq R_{2k}^\gamma(G)$ .*

*Proof.* By assumption, for any irreducible representation  $\rho$  of  $\Gamma$ , there exists some  $\eta \in R^+(\Gamma)$  such that  $\rho = \text{Res}_\Gamma^G \eta$ . Now, restriction clearly commutes with the taking of exterior powers; in other words,  $\text{Res}_\Gamma^G$  is a  $\lambda$ -ring homomorphism. Thus,

$$\begin{aligned} \sigma_{i, \rho} &= \gamma^i(\rho - \varepsilon(\rho)) = \gamma^i(\text{Res}_\Gamma^G[\eta - \varepsilon(\eta)]) \\ &= \text{Res}_\Gamma^G[\gamma^i(\eta - \varepsilon(\eta))] = \text{Res}_\Gamma^G[\sigma_{i, \eta}]. \end{aligned}$$

Consider

$$\begin{aligned} \text{Ind}_F^G(\sigma_{i_1, \rho_1} \cdots \sigma_{i_n, \rho_n}) &= \text{Ind}_F^G(\text{Res}_F^G[\sigma_{i_1, \eta_1} \cdots \sigma_{i_n, \eta_n}]) \\ &= \sigma_{i_1, \eta_1} \cdots \sigma_{i_n, \eta_n} \cdot \text{Ind}_F^G(1), \end{aligned}$$

using reciprocity. Now  $\text{Ind}_F^G(1)$  is an element of  $R_{2(r_0)}^\gamma(G) = R(G)$ , and so the monomial above has weight at least  $\sum_{j=1}^n i_j \geq k$  in the  $\gamma$ -filtration on  $R(G)$ . ■

The other maximal subgroup  $L \times \langle A_2 \rangle < H$  is abelian, and so

$$R_{2k}^\gamma(L \times \langle A_2 \rangle) = I(L \times \langle A_2 \rangle)^k = \sum_{i=0}^k \Psi_L[I(L)^i] \cdot \Psi_{\langle A_2 \rangle}[I(\langle A_2 \rangle)^{k-i}].$$

Elements of  $R_{2k}^\gamma(L \times \langle A_2 \rangle)$  are multiples of  $a_2$ ; the image of such an element under  $\Psi_{\langle A_2 \rangle}$  can be expressed as the restriction of essentially the same element from  $H$ . By reciprocity, therefore, we need only show

$$\text{Ind}_{L \times \langle A_2 \rangle}^H \circ \Psi_L(R_{2k}^\gamma(L)) \subseteq R_{2k}^\gamma(H).$$

This can be done by noting that the diagram

$$\begin{array}{ccc} R(D^1) & \xrightarrow{\Psi_{D^1}} & R(H) \\ \text{Ind} \uparrow & & \uparrow \text{Ind} \\ R(L) & \xrightarrow{\Psi_L} & R(L \times \langle A_2 \rangle) \end{array}$$

commutes, and three out of the four maps preserve the  $\gamma$ -filtration. Thus,  $H$  itself has admissible  $\gamma$ -filtration.

To show that the induction condition holds on  $H \leq D^2$ , by the same reasoning as above, it suffices to show

$$\text{Ind}_H^{D^2} \circ \Psi_{D^1}(R_{2k}^\gamma(D^1)) \subseteq R_{2k}^\gamma(D^2).$$

Because the  $\gamma$ -filtration coincides with the topological filtration for  $k \in \{0, 1, 2\}$ , we may assume  $k \geq 3$ . Explicit generators of the  $\gamma$ -filtration on  $R(D^1)$  modulo the  $\gamma$ -filtration on  $R(V^1)$  for  $k \geq 3$  may be computed directly to be

$$\begin{aligned} 2^{l-1}a_1 b_1, 2^{k-1}g_1, & \quad k = 2l + 1 \\ 2^{l-1}a_1 b_1, 2^{l-2}a_1 b_1 - 2^{k-1}g_2, & \quad k = 2l. \end{aligned}$$

The images of these elements under  $Ind_H^{D^2} \circ \varphi_{D^1}$  are

$$\begin{aligned} 2^{l-1}a_1 b_1(2 + b_2), 2^{k-1}g_2, & \quad k = 2l + 1 \\ 2^{l-1}a_1 b_1(2 + b_2), 2^{l-1}a_1 b_1 + 2^{l-2}a_1 b_1 b_2 - 2^{k-2}g_2, & \quad k = 2l. \end{aligned}$$

To show that these elements belong in the appropriate filtration ideals in  $R(D^2)$  we may use brute force for  $k = 4$  and  $5$ . The case  $k > 5$  follows by induction in jumps of  $4$ , via multiplication by the top-weight generator of the  $\gamma$ -filtration on  $R(D^2)$ , namely  $\sigma_{4,2}$  (and reduction modulo  $R_{2(\cdot)}^\gamma(V)$ ).

Next, we investigate  $J \equiv \langle D^1, A_2 B_2 \rangle$ . Because this is not a direct product, it is vastly more complicated to determine the  $\gamma$ -filtration on the representation ring. In the end, however, a nice recursive pattern emerges, again in jumps of  $4$ . More precisely,  $J$  has two irreducible representations, obtained by inducing from  $\langle B_1, A_2 B_2 \rangle$  the representation

$$\rho(y) = \begin{cases} i = \sqrt{-1}, & y = A_2 B_2 \\ 1, & y = B_1, \end{cases}$$

and its cube,  $\rho^3$ . Then it can be shown that for  $k > 4$ ,

$$R_{2k}^\gamma(J)/R_{2k}^\gamma(V_J) = [\sigma_{2, Ind \rho}]^2 \cdot R_{2(k-4)}^\gamma(J)/R_{2k}^\gamma(V_J),$$

where  $V_J = J/Z(J)$ . The maximal subgroups of  $J$  are  $D^1$  and abelian subgroups of the form  $\langle \Gamma, A_2 B_2 \rangle$ , where  $\Gamma < D^1$ . Such subgroups have admissible  $\gamma$ -filtration; direct calculation shows that the induction condition is satisfied for both.

Finally, the induction condition holds for  $J < D^2$  by virtue of the fact that  $[\sigma_{2, Ind \rho}]^2$  can be written as the restriction of an element  $z$  of weight  $4$  from the  $\gamma$ -filtration on  $R(D^2)$ . Hence, the inductive step is supplied by reciprocity;

$$\begin{aligned} Ind_J^{D^2}([\sigma_{2, Ind \rho}]^2 \cdot R_{2(k-4)}^\gamma(J)) &= z \cdot Ind_J^{D^2}(R_{2(k-4)}^\gamma(J)) \\ &\subseteq R_{2(4)}^\gamma(D^2) \cdot R_{2(k-4)}^\gamma(D^2) \subseteq R_{2k}^\gamma(D^2). \end{aligned}$$

We have proved

**THEOREM 7.2.** *The  $\gamma$ -filtration on  $D^2$  is admissible.*

### 8. COUNTER-EXAMPLES

The admissibility of the  $\gamma$ -filtration on  $D^1$  and  $D^2$  offers some cause for optimism regarding the general question of the admissibility of the



$\gamma$ -filtration on the category of  $p$ -groups. However, a counter-example is found at the next simplest extra-special 2-group,  $D^3$ .

**THEOREM 8.1.** *Let  $H = D^2 \times \langle A_3 \rangle < D^3$ . Then*

$$\text{Ind}_H^{D^3}(R_{2(4)}^\gamma(H)) \not\subseteq R_{2(4)}^\gamma(D^3).$$

*Proof.* In particular, we will show that

$$\text{Ind}_H^{D^3}(\sigma_{4,2} \otimes 1) \notin R_{2(4)}^\gamma(D^3),$$

where  $\sigma_{4,2}$  is the top-weight generator of the  $\gamma$ -filtration on  $R(D^2)$ . Consider

$$\begin{aligned} \text{Ind}_H^{D^3}(\sigma_{4,2} \otimes 1) &= \text{Ind}_H^{D^3}([-2g_2 + f_2] \otimes 1) \\ &= -2 \text{Ind}_H^{D^3}(\text{Ind}_{H_2}^{\xi_2} \otimes 1 - 4 \otimes 1) + \text{Ind}_H^{D^3}(f_2 \otimes 1) \\ &= -2 \text{Ind}_{H_2}^{\xi_3} + (f_2 + 8) \text{Ind}_H^{D^3}(1) \\ &\equiv -2g_3 + f_2(2 + b_3) \pmod{R_{2(4)}^\gamma(V)}. \end{aligned}$$

**LEMMA 8.2.**  $R_{2(4)}^\gamma(D^3)/R_{2(4)}^\gamma(V)$  has  $\mathbb{Z}$ -basis

$$\{-2g_3 + t_3, 4g_3\}.$$

*Proof.* From the results in Sections 5 and 6, we may compute the higher-weight  $\gamma$ -filtration generators to be

$$\begin{aligned} \sigma_{1,3} &= g_3, & \sigma_{2,3} &= f_3 - 7g_3, \\ \sigma_{3,3} &= 28g_3 - 6f_3, & \sigma_{4,3} &= 16f_3 - t_3 + 70g_3. \end{aligned}$$

For  $k \geq 5$ , we write the generators in terms of  $s_3 = t_3 + f_3$  and  $d_3 = f_3 - t_3$  (as it can be shown that in this case, all the coefficients will be integers);

$$\sigma_{k,3} = K_0(k, 3) \cdot g_3 + K_1(k, 3) \cdot s_3 + K_2(k, 3) \cdot d_3.$$

By Lemma 6.5, we know that  $f_3$  and  $t_3$  each have  $V$ -weight 3,  $s_3$  has  $V$ -weight 6, and  $d_3$  has  $V$ -weight 4. We may therefore reduce these generators to

$$\begin{aligned} \sigma_{2,3} &\equiv -7g_3 \pmod{R_{2(2)}^\gamma(V)} \\ \sigma_{3,3} &\equiv 28g_3 \pmod{R_{2(3)}^\gamma(V)} \\ \sigma_{4,3} &\equiv -t_3 + 70g_3 \pmod{R_{2(4)}^\gamma(V)} \\ \sigma_{k,3} &\equiv K_0(k, 3) g_3 \pmod{R_{2(4)}^\gamma(V)}, \text{ for } k \geq 5. \end{aligned}$$

A  $\mathbb{Z}$ -basis for  $R_{2(4)}^\gamma(D^3)/R_{2(4)}^\gamma(V)$  consists of monomials of the form

$$\sigma_{i_1,3} \cdot \sigma_{i_2,3} \cdots \sigma_{i_n,3} \cdot v,$$

where  $v \in R_{2k}^\gamma(V)$  ( $v$  is possibly trivial) and  $\sum i_j \geq \max\{4-k, 1\}$ . If  $v$  is non-trivial, then by Lemma 6.7,

$$\begin{aligned} \sigma_{i_j,3} \cdot v &\equiv K_2(i_j, 3) \cdot d_3 v \pmod{R_{2(k+i_j)}^\gamma(V)} \\ &\equiv 0 \pmod{R_{2(4)}^\gamma(V)}. \end{aligned}$$

Hence, we may assume that  $v = 1$ . Given the list of reduced generators and the argument above applied to  $t_3 \in R_{2(3)}^\gamma(V)$ , we have for  $n > 1$

$$\begin{aligned} \sigma_{i_1,3} \cdots \sigma_{i_n,3} &= \left[ \prod_{j=1}^n K_0(i_j, 3) \right] \cdot g_3^n \pmod{R_{2(4)}^\gamma(V)} \\ &= \pm \left[ \prod_{j=1}^n K_0(i_j, 3) \right] \cdot 2^{4n} g_3 \pmod{R_{2(4)}^\gamma(V)}, \end{aligned}$$

using the fact that

$$g_3^2 = s_3 - 16g_3 \equiv -16g_3 \pmod{R_{2(4)}^\gamma(V)}.$$

But  $\sigma_{8,3} \equiv -16g_3 \pmod{R_{2(4)}^\gamma(V)}$ , and so all monomials having  $n > 1$  are redundant. This shows that  $R_{2(4)}^\gamma(D^3)/R_{2(4)}^\gamma(V)$  is generated over  $\mathbb{Z}$  by

$$\{-t_3 + 70g_3, -16g_3\};$$

we simplify this basis to the one in the statement of the lemma by noting

$$\begin{aligned} 3(-t_3 + 70g_3) + 13(-16g_3) &= -3t_3 + 2g_3 \equiv -t_3 + 2g_3 \pmod{R_{2(4)}^\gamma(V)}, \\ 2(-t_3 + 70g_3) + 9(-16g_3) &= -2t_3 - 4g_3 \equiv -4g_3 \pmod{R_{2(4)}^\gamma(V)}. \quad \blacksquare \end{aligned}$$

The proof of Proposition 8.1 now follows quickly. Consider the expansions

$$\begin{aligned} t_3 &\equiv 2a_1 b_1 + 2a_2 b_2 + 2a_3 b_3 \pmod{R_{2(4)}^\gamma(V)}, \\ f_2(2 + b_3) &\equiv 2a_1 b_1 + 2a_2 b_2 + a_2 b_2 b_3 + a_1 b_1 b_3 \pmod{R_{2(4)}^\gamma(V)}. \end{aligned}$$

Ignoring the  $g_3$  term, it is clear that the  $I(V)$  portion of  $\text{Ind}_H^{D^3}(\sigma_{4,2} \otimes 1)$ , namely  $f_2(2 + b_3)$ , cannot be written as a multiple of  $-t_3 + 2g_3$ .  $\blacksquare$

We would like to generalize this result to provide a counter-example to the admissibility of the  $\gamma$ -filtration on  $D^r$  for any  $r \geq 3$ . If we let  $H_r = D^{r-1} \times \langle A_r \rangle$ , it seems reasonable to hope that the induction

condition fails on each  $H_r < D^r$ ; in particular, we would like to show that, as in the case  $r = 3$ ,

$$\text{Ind}_{H_r}^{D^r}(\sigma_{2^{r-1}, r-1} \otimes 1) \notin R_{2(2^{r-1})}^\gamma(D^r).$$

Now, from Lemma 6.1, we know that the top-weight generator of the  $\gamma$ -filtration on  $D^{r-1}$  is

$$\sigma_{2^{r-1}, r-1} = (-2^{2^{r-1}-r})g_{r-1} + (2^{2^{r-1}-2r+1})s_{r-1} + (2^{2^{r-2}-r+1})d_{r-1}.$$

For  $r > 3$ , these coefficients are all integers. It is easy to work out

$$\begin{aligned} \text{Ind}_{H_r}^{D^r}(\sigma_{2^{r-1}, r-1} \otimes 1) &= (-2^{2^{r-1}-r})(g_r - 2^{r-1}b_r) + (2^{2^{r-1}-2r+1}s_{r-1} + 2^{2^{r-2}-r}d_{r-1})(2 + b_r) \\ &\equiv -2^{2^{r-1}-r}g_r + (2^{2^{r-2}-r}d_{r-1})(2 + b_r) \pmod{R_{2(2^{r-1})}^\gamma(V)}. \end{aligned}$$

Mimicking the proof of Proposition 8.1, our aim is to show that the  $I(V)$  portion of this expression, namely  $2^{2^{r-2}-r}d_{r-1}(2 + b_r)$ , cannot appear in  $R_{2(2^{r-1})}^\gamma(D^r)$ , with or without attached  $g_r$  terms.

First consider monomials of the form

$$\sigma_{i_1, r} \cdots \sigma_{i_n, r},$$

where  $\sum_j i_j \geq 2^{r-1}$  and each  $i_j > 1$ . If we express each  $\sigma_{i_j, r}$  in terms of  $g_r$ ,  $s_r$ , and  $d_r$ , then by the list of products (6.6), we see that

$$\sigma_{i_1, r} \cdots \sigma_{i_n, r} = \mathcal{M}_0 g_r + \mathcal{M}_1 s_r + \mathcal{M}_2 d_r,$$

for some  $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \in \mathbb{Z}[1/2]$ . In other words, the only elements of  $I(V)$  potentially appearing in filtrations beyond their  $V$ -weights are  $\mathbb{Z}[1/2]$ -linear combinations of  $s_r$  and  $d_r$ . Clearly,  $2^{2^{r-2}-r}d_{r-1}(2 + b_r)$  does not fall into this set.

We move to the case of monomials having a non-trivial factor in  $I(V)$ ,

$$\sigma_{i_1, r} \cdots \sigma_{i_n, r} \cdot v,$$

where  $v \in R_{2k}^\gamma(V)$  for some  $k \geq 1$  and  $k + \sum_j i_j \geq 2^{r-1}$ . By Lemma 6.7, we have

$$\begin{aligned} \sigma_{i_1, r} \cdots \sigma_{i_n, r} \cdot v &\equiv \left( \prod_{j=1}^n K_2(i_j, r) \right) d_r^n \cdot v \pmod{R_{2(2^{r-1})}^\gamma(V)} \\ &\equiv \left( \prod_{j=1}^n K_2(i_j, r) \right) [(-2)^{r+1}]^{n-1} d_r v \pmod{R_{2(2^{r-1})}^\gamma(V)}. \end{aligned}$$

Now, by considering characters, we have

$$d_r \cdot d_{r-1} = -2^{r+1}d_{r-1}.$$

In particular, if we let  $v = d_{r-1}(2 + b_r)$ , which has  $V$ -weight  $r + 1$ , then

$$\begin{aligned} \sigma_{i_1, r} \cdots \sigma_{i_n, r} \cdot d_{r-1}(2 + b_r) \\ \equiv \pm \left( \prod_{j=1}^n K_2(i_j, r) \right) 2^{(r+1)n} d_{r-1}(2 + b_r) \pmod{R_{2(2^r-1)}^y(V)}. \end{aligned}$$

For  $2^{2^{r-2}-r}d_{r-1}(2 + b_r)$  to lie in  $R_{2(2^r-1)}^y(D^r)$ , we must find some  $\{i_1, \dots, i_n\}$  such that  $\sum_j i_j \geq 2^{r-1} - (r + 1)$  and

$$(r + 1)n + v_2 \left[ \prod_{j=1}^n K_2(i_j, r) \right] \leq 2^{r-2} - r.$$

The key now lies in being able to get a handle on

$$\begin{aligned} v_2[K_2(k, r)] &= v_2[(-1)^k 2^r \wp_{\text{odd}}(k, r)] \\ &= v_2 \left[ \sum_{\substack{l=1 \\ l \text{ odd}}}^{\lfloor k/2 \rfloor} \binom{2^r - 2l}{k - 2l} \binom{2^{r-1}}{l} \right] - r. \end{aligned}$$

This we do through a series of lemmas.

LEMMA 8.3. For any  $1 \leq k \leq 2^r$ ,

$$v_2 \left[ \sum_{l=0}^{\lfloor k/2 \rfloor} \binom{2^{r-1}}{l} \binom{2^r - 2l}{k - 2l} \right] \geq \lfloor k/2 \rfloor + r - v_2(k),$$

with equality holding if  $k \not\equiv 2 \pmod{4}$ .

*Proof.* For simplicity, let  $m = 2^{r-1}$ . The sum in question appears as the coefficient of  $X^k$  in the generating function

$$Q_m(X) = [X^2 + (X + 1)^2]^m = \sum_{\lambda=0}^{2m-2l} \sum_{l=0}^m \binom{m}{l} \binom{2m-2l}{\lambda} X^{2l+\lambda}.$$

We can rewrite this sum by noting that

$$\begin{aligned} Q_m(X) &= [2X^2 + 2X + 1]^m = [2X(X + 1) + 1]^m \\ &= \sum_{l=0}^m \binom{m}{l} 2^l X^l (X + 1)^l \\ &= \sum_{l=0}^m \sum_{\mu=0}^l \binom{m}{l} \binom{l}{\mu} 2^l X^{l+\mu}. \end{aligned}$$

Our task is thus to determine the 2-adic absolute value of

$$C(m, k) = \sum_{l=\lfloor k/2 \rfloor}^k \binom{m}{l} \binom{l}{k-l} 2^l.$$

We now divide into two cases.

If  $k = 2n + 1$ , let  $j = l - n$  to get

$$\begin{aligned} C(m, 2n + 1) &= \sum_{j=0}^n \binom{m}{n+j+1} \binom{n+j+1}{n-j} 2^{n+j+1} \\ &= \sum_{j=0}^n \frac{m! 2^{n+j+1}}{(m-n-j-1)! (n-j)! (2j+1)!} \\ &= \sum_{j=0}^n A(m, 2n + 1, j). \end{aligned}$$

We compare the 2-adic absolute values of each term in this sum. For this, recall from the proof of Sublemma 6.4 that for any positive integer  $a$ ,  $v_2(a!) = a - \beta(a)$ , where  $\beta(a)$  denotes the number of ones in the binary expansion of  $a$ . Properties of  $\beta$  include  $\beta(2a) = \beta(a)$  and  $\beta(2a + 1) = \beta(a) + 1$ ; less obvious, but well-known among combinatorialists, is the fact that  $\beta(a) + \beta(b) - \beta(a + b)$  is equal to the number of carries that occur when  $a$  is added to  $b$  in base 2. Hence,

$$\begin{aligned} v_2[A(m, 2n + 1, j)] &= n + j + 1 + m - \beta(m) - (m - n - j - 1) \\ &\quad + \beta(m - n - j - 1) \\ &\quad - (n - j) + \beta(n - j) - (2j + 1) + \beta(2j + 1) \\ &= n + 2 + [j - \beta(j)] + [\beta(m - n - j - 1) + \beta(j)] \\ &\quad + [\beta(n - j) + \beta(j)] - \beta(m) \\ &\geq n + 2 + \beta(m - n - 1) + \beta(n) - \beta(m) \\ &= v_2[A(m, 2n + 1, 0)]. \end{aligned}$$

Equality holds if and only if  $j = \beta(j)$  and no carries occur when adding  $j$  either to  $m - n - j - 1$  or to  $n - j$ . Now,  $j = \beta(j)$  only when  $j = 0$  or 1. If  $j = 1$ , then in order for no carries to occur when adding 1 to  $n - 1$ ,  $n$  must be odd; if in addition no carries occur when adding 1 to  $m - n - 2$ ,  $m$  must be odd. Since  $m = 2^l$  is even, the inequality is strict for  $j > 0$ . This implies that

$$\begin{aligned} v_2[C(m, 2n + 1)] &= v_2[A(m, 2n + 1, 0)] \\ &= n + 2 + \beta(m - n - 1) + \beta(n) - \beta(m). \end{aligned}$$

Recall that  $m = 2^{r-1}$ ; then  $m - 1$  has 2-adic expansion

$$1 + 2 + 2^2 + \cdots + 2^{r-2};$$

from this it is easy to see that  $\beta(m - n - 1) + \beta(n) = r - 1$  for  $n \leq 2^{r-1}$ . The conclusion of the lemma now follows.

Now we consider the case  $k = 2n$ . Again setting  $j = l - n$ , we get

$$\begin{aligned} C(m, 2n) &= \sum_{j=0}^{n+1} \binom{m}{n+j} \binom{n+j}{n-j} 2^{n+j} \\ &= \sum_{j=0}^{n+1} \frac{m! 2^{n+j}}{(m-n-j)! (n-j)! (2j)!} = \sum_{j=0}^n A(m, 2n, j). \end{aligned}$$

Using the same arguments as above, we obtain

$$\begin{aligned} v_2[A(m, 2n, j)] &= n + j + m - \beta(m) - (m - n - j) + \beta(m - n - j) \\ &\quad - (n - j) + \beta(n - j) - (2j) + \beta(2j) \\ &= n + [j - \beta(j)] + [\beta(m - n - j) + \beta(j)] \\ &\quad + [\beta(n - j) + \beta(j)] - \beta(m) \\ &\geq n + \beta(m - n) + \beta(n) - \beta(m) = v_2[A(m, 2n, 0)], \end{aligned}$$

with equality holding if and only if  $j = 0$  or  $1$  and no carries occur when adding  $j$  either to  $m - n - j$  or to  $n - j$ . In particular, we get equality when  $j = 1$  and  $n$  is odd. Thus,  $v_2[C(m, 2n)] = v_2[A(m, 2n, 0)]$  when  $k \equiv 0 \pmod{4}$  and  $v_2[C(m, 2n)] \geq v_2[A(m, 2n, 0)] + 1$  when  $k \equiv 2 \pmod{4}$ . If the 2-adic expansion of  $n$  is

$$(1 + a_1 2 + a_2 2^2 + \cdots + a_s 2^s) 2^t,$$

then the 2-adic expansion of  $m - n = 2^{r-1} - n$  is

$$2^t + \sum_{i=1}^s (1 - a_i) 2^{i+t} + 2^{s+t+1} + \cdots + 2^{r-2},$$

from which it is clear that  $\beta(2^{r-1} - n) = r - \beta(n) - v_2(n)$ . This quickly completes the proof of the lemma.

**COROLLARY 8.4.** For all  $k \leq 2^r$ ,

$$v_2[K_2(k, r)] \geq \lfloor k/2 \rfloor - v_2(k) - 1.$$

*Proof.* In the proof of Lemma 6.2, we saw that

$$\sum_{l=0}^{\lfloor k/2 \rfloor} (-1)^l \binom{2^{r-1}}{l} \binom{2^r - 2l}{k - 2l} = 2^k \binom{2^{r-1}}{k},$$

which has 2-adic absolute value  $k - 1 + r - v_2(k)$ . For  $k \geq 3$ , this is strictly greater than the 2-adic absolute value of the non-alternating sum. Now use the fact that

$$2 \sum_{l \text{ odd}} (-) = \sum_{\text{all } l} (-) - \sum_{\text{all } l} (-1)^l (-).$$

The cases  $k = 1, 2$  can be established by direct calculation.

**PROPOSITION 8.5.** *For all  $r \geq 3$ , the  $\gamma$ -filtration on  $D^r$  is not admissible, as*

$$\text{Ind}_{H_r}^{D^r}(\sigma_{2^{r-1}, r-1} \otimes 1) \notin R_{2(2^{r-1})}^\gamma(D^r).$$

*Proof.* Let  $\{i_1, \dots, i_n\}$  be such that  $\sum_j i_j \geq 2^{r-1} - r - 1$ . Then

$$\begin{aligned} & (r+1)n + v_2 \left[ \prod_{j=1}^n K_2(i_j, r) \right] \\ &= rn + n + \sum_{j=1}^n v_2[K_2(i_j, r)] \\ &= rn + n + \sum_{i_j \text{ even}}^{m_0} v_2[K_2(i_j, r)] + \sum_{i_j \text{ odd}}^{m_1} v_2[K_2(i_j, r)] \\ &\geq rn + n + \sum_{i_j \text{ even}}^{m_0} (\frac{1}{2}i_j - v_2(i_j) - 1) + \sum_{i_j \text{ odd}}^{m_1} (\frac{1}{2}(i_j - 1) - 1) \\ &= rn + n + \frac{1}{2} \sum_{j=1}^n i_j - \sum_{i_j \text{ even}}^{m_0} v_2(i_j) - m_0 - \frac{m_1}{2} - m_1 \\ &\geq rn + \frac{1}{2}(2^{r-1} - r - 1) - rm_0 - \frac{m_1}{2} \\ &= 2^{r-2} + r(n - m_0) - \frac{1}{2}(r + 1 + m_1) \\ &= 2^{r-2} + rm_1 - \frac{1}{2}(r + 1 + m_1). \end{aligned}$$

It suffices now to show that this quantity is strictly greater than  $2^{r-2} - r$ ; this is true if and only if

$$\begin{aligned} & rm_1 - \frac{1}{2}(r + 1 + m_1) > -r \\ &\Leftrightarrow 12r - 1r(1 + m_1) > \frac{1}{2}(r + 1 + m_1) \\ &\Leftrightarrow (2r - 1)(1 + m_1) > r. \end{aligned}$$

But this last inequality obviously holds, as  $m_1 \geq 0$  and  $2r - 1 > r$  for  $r > 3$ . ■

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