Posets of geometric graphs

Debra L. Boutin

Department of Mathematics, Hamilton College, Clinton, NY 13323, USA

Sally Cockburn

Department of Mathematics, Hamilton College, Clinton, NY 13323, USA

Alice M. Dean

Mathematics and Computer Science Department, Skidmore College
Saratoga Springs, NY 12866, USA

Andrei M. Margea

Department of Computer Science, University of Texas, Austin, TX 78712, USA

Abstract

A *geometric graph* $\overline{G}$ is a simple graph drawn in the plane, on points in general position, with straight-line edges. We call $\overline{G}$ a *geometric realization* of the underlying abstract graph G. A *geometric homomorphism* $f : \overline{G} \to \overline{H}$ is a vertex map that preserves adjacencies and crossings (but not necessarily non-adjacencies or non-crossings). This work uses geometric homomorphisms to introduce a partial order on the set of isomorphism classes of geometric realizations of an abstract graph G. Set $\overline{G} \preceq \widehat{G}$ if $\overline{G}$ and $\widehat{G}$ are geometric realizations of G and there is a vertex-injective geometric homomorphism $f : \overline{G} \to \widehat{G}$. This paper develops tools to determine when two geometric realizations are comparable. Further, for $3 \leq n \leq 6$, this paper provides the isomorphism classes of geometric realizations of P_n, C_n and K_n, as well as the Hasse diagrams of the geometric homomorphism posets (resp., $\mathcal{P}_n, \mathcal{C}_n, \mathcal{K}_n$) of these graphs. The paper also provides the following results for general n: each of $\mathcal{P}_n$ and $\mathcal{C}_n$ has a unique minimal element and a unique maximal element; if $k \leq n$ then $\mathcal{P}_k$ (resp., $\mathcal{C}_k$) is a subposet of $\mathcal{P}_n$ (resp., $\mathcal{C}_n$); and $\mathcal{K}_n$ contains a chain of length $n-2$.

Keywords: Geometric graph, homomorphism, poset.

Math. Subj. Class.: 05C62, 06A06

E-mail addresses: dboutin@hamilton.edu (Debra L. Boutin), scockbur@hamilton.edu (Sally Cockburn), adean@skidmore.edu (Alice M. Dean), utistu87@cs.utexas.edu (Andrei M. Margea)

1 Introduction

The topic of graph homomorphisms has been a subject of growing interest; for an excellent survey of the area see [9]. In [3], Boutin and Cockburn extend the theory of graph homomorphisms to geometric graphs. In this paper we use geometric homomorphisms to define and study posets of geometric realizations of a given abstract graph. Throughout this work the term graph means simple graph.

A geometric graph $\overline{G}$ is a graph drawn in the plane, on points in general position, with straight-line edges. What we care about in a geometric graph is which pairs of vertices are adjacent and which pairs of edges cross. Thus two geometric graphs will be called isomorphic if there is a bijection between their vertex sets that preserves adjacencies, non-adjacencies, crossings, and non-crossings. Note that there are other ways to define geometric graph isomorphism; this is discussed further in Section 2.

In [3], the definition of graph homomorphism was extended to the context of geometric graphs in the following way. A geometric homomorphism is a vertex map $f : \overline{G} \to \overline{H}$ that preserves both adjacencies and crossings (but not necessarily non-adjacencies or non-crossings). If such a map exists we write $\overline{G} \to \overline{H}$ and say that $\overline{G}$ is (geometrically) homomorphic to $\overline{H}$. There are many similarities between abstract graph homomorphisms and geometric graph homomorphisms, but there are also great contrasts. Results that are straightforward in the context of abstract graphs can be complex in the context of geometric graphs.

In abstract graph homomorphism theory, two vertices cannot be identified under any homomorphism if and only if they are adjacent. However, in [3] we see that there are additional reasons why two vertices cannot be identified under geometric homomorphism: if they are involved in a common edge crossing; if they are endpoints of an odd length path each edge of which is crossed by a common edge; if they are endpoints of a path of length two whose edges cross all edges of an odd length cycle. This list is likely not exhaustive.

In abstract graph homomorphism theory, a graph is not homomorphic to a graph on fewer vertices if and only if it is a complete graph. In geometric homomorphism theory, there are many graphs other than complete graphs that are not homomorphic to any geometric graph on fewer vertices. For example, since vertices involved in a common crossing cannot be identified by geometric homomorphism, there is no geometric homomorphism of the non-plane realization of C_4 into a geometric graph on fewer than four vertices.

In abstract graph homomorphism theory, every graph on n vertices is homomorphic to K_n. However, not all geometric graphs on n vertices are homomorphic to a given realization of K_n. In fact, two different geometric realizations of K_n are not necessarily homomorphic to each other. For example, consider the three geometric realizations of K_6 given in Figure 1. Note that $\overline{G}_2$ has a vertex with all incident edges crossed, while $\overline{G}_3$ does not; this can be used to prove that there is no geometric homomorphism from $\overline{G}_2$ to $\overline{G}_3$. Also, $\overline{G}_3$ has more crossings than $\overline{G}_2$; this can be used to prove that there is no geometric homomorphism from $\overline{G}_3$ to $\overline{G}_2$. On the other hand we can easily argue that while there is no geometric homomorphism from $\overline{G}_2$ to $\overline{G}_1$, the map $f : \overline{G}_1 \to \overline{G}_2$ implied by the given vertex numbering schemes is a geometric homomorphism.

Since homomorphisms are reflexive and transitive, it is natural to want to use them to induce a partial order. That is, we would like to define $\overline{G} \preceq \overline{H}$ when $\overline{G} \to \overline{H}$. However, graph homomorphisms are not necessarily antisymmetric. It is easy to find geometric (or abstract) graphs $\overline{G}$ and $\overline{H}$ so that $\overline{G} \to \overline{H}$ and $\overline{H} \to \overline{G}$ but $\overline{G}$ and $\overline{H}$ are not isomorphic. For example, let $\overline{H}$ be any geometric graph with a non-isolated vertex z. Add a vertex x
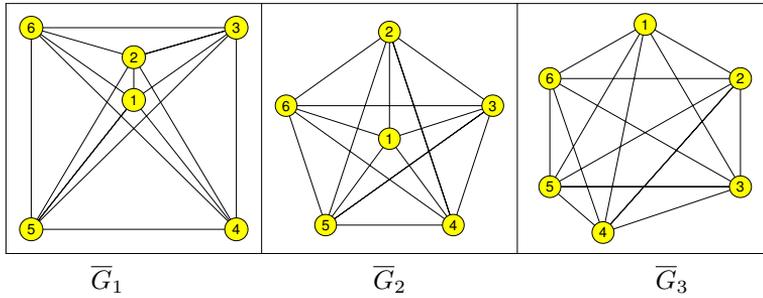
Figure 1: Three geometric realizations of K_6

and edge e between x and z, positioned so that e crosses no other edge of $\overline{H}$. Call this new graph $\overline{G}$. Identifying x with any neighbor of z gives us $\overline{G} \rightarrow \overline{H}$. The fact that $\overline{H}$ is a subgraph of $\overline{G}$ gives us $\overline{H} \rightarrow \overline{G}$. But clearly $\overline{G}$ and $\overline{H}$ are not isomorphic. Thus graph homomorphisms (whether abstract or geometric) do not induce a partial order since they are not antisymmetric.

In [9], Hell and Nešetřil solve this problem for abstract graphs by using homomorphisms to define a partial order on the class of non-isomorphic cores of graphs. The *core* of an (abstract or geometric) graph is the smallest subgraph to which it is homomorphic. In the example discussed above, $\overline{G}$ and $\overline{H}$ have isomorphic cores. In this paper we solve the problem by using geometric homomorphisms to define a partial order on the set of geometric realizations of a given abstract graph. That is to say, we let $\overline{G} \preceq \overline{H}$ if there is a geometric homomorphism $f : \overline{G} \rightarrow \overline{H}$ that induces an isomorphism on the underlying abstract graphs. This definition ensures that $\preceq$ is antisymmetric.

The paper is organized as follows. In Section 2 we give formal definitions and develop tools that help determine whether two geometric realizations are homomorphic. In Section 3 we determine the isomorphism classes and resulting poset, $\mathcal{P}_n$, of realizations of the path P_n with $2 \leq n \leq 6$. Additionally we provide the following results: $\mathcal{P}_n$ has a unique minimal and a unique maximal element; if $k \leq n$, then $\mathcal{P}_k$ is a subposet of $\mathcal{P}_n$; for each positive integer c less than or equal to the maximum number of crossings, there is at least one realization of P_n with precisely c crossings. In Section 4 we determine the isomorphism classes and resulting poset, $\mathcal{C}_n$, of realizations of the cycle C_n with $3 \leq n \leq 6$. We also show that $\mathcal{C}_n$ has a unique minimal element and a unique maximal element. Further if $k \leq n$, then $\mathcal{C}_k$ is a subposet of $\mathcal{C}_n$. In Section 5 we determine the isomorphism classes and resulting poset, $\mathcal{K}_n$, of realizations of the complete graph K_n with $3 \leq n \leq 6$, and we prove that for all n, $\mathcal{K}_n$ contains a chain of length $n - 2$. In Section 6 we provide some open questions.

2   Basics, tools, examples

A *geometric graph* $\overline{G}$ is a simple graph $G = \big(V(G), E(G)\big)$ together with a straight-line drawing of G in the plane with vertices in general position (no three vertices are collinear and no three edges cross at a single point). A geometric graph $\overline{G}$ with underlying abstract graph G is called a *geometric realization* of G; the term *rectilinear drawing* is also used in the literature. Two geometric realizations of a graph are considered the same if they have the same vertex adjacencies and edge crossings. This is formalized below by extending the

definition of graph isomorphism in a natural way to geometric graphs.

Definition. [1] A geometric isomorphism, denoted $f : \overline{G} \to \overline{H}$, is a bijection $f : V(\overline{G}) \to V(\overline{H})$ such that for all $u, v, x, y \in V(\overline{G})$,

1. $uv \in E(\overline{G})$ if and only if $f(u)f(v) \in E(\overline{H})$, and

2. xy crosses uv in $\overline{G}$ if and only if $f(x)f(y)$ crosses $f(u)f(v)$ in $\overline{H}$.

If there exists a geometric isomorphism $f : \overline{G} \to \overline{H}$, we write $\overline{G} \cong \overline{H}$. Geometric isomorphism clearly defines an equivalence relation on the set of geometric realizations of a simple graph G.

Note that in [7, 8] Harborth, et al., give a definition for isomorphism of geometric graphs that is stricter than the one given here. They require that a geometric isomorphism also preserve regions and parts of edges. Figure 2 shows two geometric realizations of C_6 that have the same crossings (and so are isomorphic by our definition) but have different regions (and so are not isomorphic in the sense of Harborth). Harborth's definition is more geometric; it allows only perturbations of vertex positions within their assigned regions. The definition used here is more combinatorial; it is a natural generalization of the definition of graph isomorphism and generalizes easily to geometric graph homomorphism. It captures facts that come from the geometry, without capturing the geometry itself. This more combinatorial definition has been used in other works about geometric graph automorphisms and geometric graph homomorphisms. See for instance [1], [2], [3], and [5].
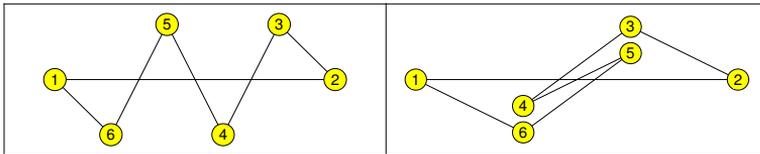

Figure 2: Realizations with the same crossings but different regions

One consequence of the differences in these definitions is that for a given abstract graph there are (potentially) fewer isomorphism classes of realizations under our definition than under that of Harborth. Thus if we have an isomorphism class of Harborth et al., and no pair of representatives is isomorphic by this definition, then we have found all isomorphism classes under this definition. This is done in Section 5 for all isomorphism classes of geometric realizations of K_3, K_4, K_5, K_6.

Just as the definition of graph isomorphism is extended to geometric graphs, so is the definition of graph homomorphism.

Definition. [3] A geometric homomorphism, denoted $f : \overline{G} \to \overline{H}$, is a function $f : V(\overline{G}) \to V(\overline{H})$ such that for all $u, v, x, y \in V(\overline{G})$,

1. if $uv \in E(\overline{G})$, then $f(u)f(v) \in E(\overline{H})$, and

2. if xy crosses uv in $\overline{G}$, then $f(x)f(y)$ crosses $f(u)f(v)$ in $\overline{H}$.

If there is a geometric homomorphism $f : \overline{G} \to \overline{H}$, we write $\overline{G} \xrightarrow{f} \overline{H}$ or simply $\overline{G} \to \overline{H}$, and we say that $\overline{G}$ is (geometrically) homomorphic to $\overline{H}$.

Definition. Let $\overline{G}$ and $\widehat{G}$ be geometric realizations of a graph G. Set $\overline{G} \preceq \widehat{G}$ (or $\overline{G} \overset{f}{\preceq} \widehat{G}$) if there exists a (vertex) injective geometric homomorphism $f : \overline{G} \to \widehat{G}$.

Note that since the abstract graphs underlying $\overline{G}$ and $\widehat{G}$ are the same, the fact that f is injective and preserves adjacency means that f induces an isomorphism from G to itself. It is not difficult to see that this relation is reflexive, transitive, antisymmetric, and hence a partial order.

Definition. The *geometric homomorphism poset* of a graph is the set of geometric isomorphism classes of its realizations partially ordered by the relation $\preceq$.

2.1 The edge crossing and line/crossing graphs

Recall that the *line graph* of an abstract graph G, denoted $L(G)$, is the abstract graph whose vertices correspond to edges of G, with adjacency when the corresponding edges of G are adjacent. In this section, we define the *edge crossing graph*, which is similar to the line graph except that it encodes edge crossings rather than edge adjacencies. In [10] Nešetřil, et al., proved a correspondence between graph homomorphisms and homomorphisms of their line graphs. We generalize this to geometric graphs and the union of their line and crossing graphs.

Definition. Let $\overline{G}$ be a geometric graph. Define the *edge crossing graph*, denoted $EX(\overline{G})$, to be the abstract graph whose vertices correspond to edges of $\overline{G}$, with adjacency when the corresponding edges of $\overline{G}$ cross. Define the *line/crossing graph*, denoted $LEX(\overline{G})$, to be the 2-edge-colored abstract graph whose vertices correspond to the edges of $\overline{G}$, with red edges in $LEX(\overline{G})$ corresponding to adjacent edge pairs in $\overline{G}$ and black (and dashed) edges in $LEX(\overline{G})$ corresponding to crossing edge pairs in $\overline{G}$.

In other words, the line/crossing graph of $\overline{G}$ is the union of the line graph of G and the edge crossing graph of $\overline{G}$, with an added edge coloring to keep the meanings of these edges clear. Figure 3 shows a geometric realization of P_6 and its line/crossing graph.

The following theorem is well-known and important to our proofs.

Theorem 2.1. [11] If G has more than four vertices, then $G \cong H \iff L(G) \cong L(H)$.

Proposition 2.2. Let $\overline{G}$ and $\overline{H}$ be geometric graphs on more than four vertices. Then $\overline{G} \preceq \overline{H}$ if and only if there exists a color-preserving graph homomorphism $\tilde{f} : LEX(\overline{G}) \to LEX(\overline{H})$ that restricts to an isomorphism from $L(G)$ to $L(H)$.

Note that using Proposition 2.2 requires that $\overline{G}$ and $\overline{H}$ have isomorphic underlying abstract graphs. Thus we may assume that $\overline{G}$ and $\overline{H}$ are geometric realizations of the same graph.

An alternate way to phrase Proposition 2.2 is to say that if $\overline{G}$ and $\widehat{G}$ are geometric realizations of the abstract graph G, then $\overline{G} \preceq \widehat{G}$ if and only if there exists an isomorphism of the line graphs that induces a homomorphism of the edge crossing graphs. Thus if there is no injective homomorphism $\tilde{f} : EX(\overline{G}) \to EX(\widehat{G})$, then $\overline{G} \npreceq \widehat{G}$. Thus given $\overline{G}$ and $\widehat{G}$, if $EX(\overline{G})$ is not isomorphic to a subgraph of $EX(\widehat{G})$, then there is no geometric homomorphism $\overline{G} \to \widehat{G}$. However, Proposition 2.2 tells us something stronger. The proposition tells us that $\overline{G} \to \widehat{G}$ if and only if there is $\tilde{f} \in \mathrm{Aut}(L(G))$ so that $\tilde{f}(EX(\overline{G}))$ is a subgraph of $EX(\widehat{G})$. For graphs whose line graphs have small automorphism groups, this reduces the work significantly.

Example. In Figure 3 we see a realization $\overline{P}_6$ of P_6. Note that $L(P_6) \cong P_5$ and $EX(\overline{P}_6) \cong P_2$. Each non-plane geometric realization of P_6 has edge crossing graph with at least one edge, so theoretically there are many possible realizations $\widehat{P}_6$ to which $\overline{P}_6$ might be geometrically homomorphic. However, by Proposition 2.2 a vertex injective homomorphism $f : \overline{P}_6 \to \widehat{P}_6$, taking $EX(\overline{P}_6)$ to a subgraph of $EX(\widehat{P}_6)$, needs to be an automorphism of $L(P_6)$. Recall that $\overline{P}_6$ as given in Figure 3 has a single crossing that occurs between edges e_1 (with endpoints 1 and 2) and e_3 (with endpoints 3 and 4). Applying the two automorphisms of $L(P_6)$ to this realization, we see that $\overline{P}_6 \to \widehat{P}_6$ if and only if $\widehat{P}_6$ has one of the crossings $e_1 \times e_3$ or $e_3 \times e_5$. This restricts our search significantly.
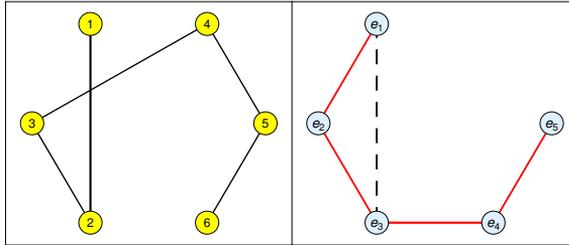
Figure 3: $\overline{P}_6$ and its line/crossing graph

2.2 Parameters

We next define parameters that help determine whether there is a geometric homomorphism between two geometric realizations of the same graph. Proposition 2.3 lists several properties that follow easily from the definition of these parameters.
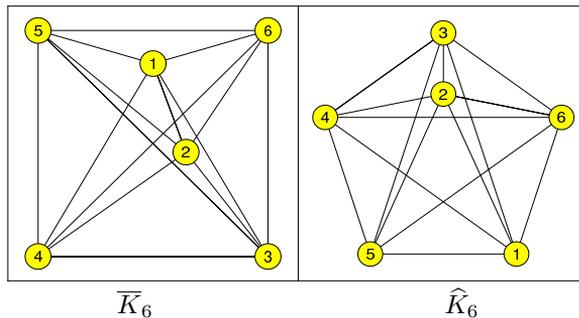

Figure 4: Vertex labels giving a geometric homomorphism from $\overline{K}_6$ to $\widehat{K}_6$

Figure 4 shows two geometric realizations, $\overline{K}_6$ and $\widehat{K}_6$, of K_6. The vertex labeling gives a geometric homomorphism from $\overline{K}_6$ to $\widehat{K}_6$. Below we define several parameters for geometric embeddings, which we then use to demonstrate that there is no geometric homomorphism from $\widehat{K}_6$ to $\overline{K}_6$. In what follows we let $e_{i,j}$ denote the edge from vertex i to vertex j.

Definition. If $\overline{G}$ is a geometric realization of a graph G, let $cr(\overline{G})$ denote the total number of crossings in $\overline{G}$. For $e \in E(\overline{G})$ let $cr(e)$ be the number of edges that cross the edge e in

$\overline{G}$. Let E_0 (resp., $E_\times$) denote the set of edges in $\overline{G}$ that have $cr(e) = 0$ (resp., $cr(e) > 0$). If $|E_\times| = 0$ we say that $\overline{G}$ is a *plane realization* of G. Let $\overline{G}_0$ (resp., $\overline{G}_\times$) denote the abstract graph on the full vertex set of G whose edge set is E_0 (resp., $E_\times$). A clique in $\overline{G}$ is called a *convex clique* if its vertices are in convex position. The *convex clique number* of $\overline{G}$ is the maximum size of a convex clique in $\overline{G}$, denoted by $\widehat{\omega}(\overline{G})$.

Proposition 2.3. Let $\overline{G}$ and $\widehat{G}$ be geometric realizations of a graph G, and suppose $\overline{G} \overset{f}{\preceq} \widehat{G}$. Then

1. $cr(\overline{G}) \le cr(\widehat{G})$.
2. For each $e \in E(\overline{G})$, $cr(e) \le cr(f(e))$.
3. $|E_0(\overline{G})| \ge |E_0(\widehat{G})|$ and $|E_\times(\overline{G})| \le |E_\times(\widehat{G})|$.
4. $\widehat{G}_0$ is a subgraph of $\overline{G}_0$.
5. $\widehat{\omega}(\overline{G}) \le \widehat{\omega}(\widehat{G})$.

Example. In the graphs in Figure 4, $cr(\overline{K}_6) = 8$, $cr(\widehat{K}_6) = 11$, $|E_0(\overline{K}_6)| = 7$, $|E_0(\widehat{K}_6)| = 6$, $\widehat{\omega}(\overline{K}_6) = 4$ and $\widehat{\omega}(\widehat{K}_6) = 5$. Thus parts 1, 3, and 5 of Proposition 2.3 each imply that there is no geometric homomorphism from $\widehat{K}_6$ to $\overline{K}_6$.

Definition. Let $\overline{G}$ be a geometric realization of a graph G. For each $v \in V(\overline{G})$, let $d_0(v)$ be the number of uncrossed edges incident to v and let $m(v)$ be the maximum number of times an edge that is incident to v is crossed.

Proposition 2.4. Let $\overline{G}$ and $\widehat{G}$ be geometric realizations of a graph G, and suppose $\overline{G} \overset{f}{\preceq} \widehat{G}$. Then for each $v \in V(\overline{G})$, $d_0(v) \ge d_0(f(v))$ and $m(v) \le m(f(v))$.

Example. In the example in Figure 4, consider the vertex $v = 1$ in $\overline{K}_6$ and its image $f(v)$ in $\widehat{K}_6$. Then $d_0(v) = 2$ and $m(v) = 2$, while $d_0(f(v)) = 2$ and $m(f(v)) = 3$.

An effective way to use Proposition 2.4 is to compare the values of each parameter over all vertices at once. This motivates the following definitions.

Definition. For a geometric graph $\overline{G}$, let $D_0(\overline{G})$ (resp., $M(\overline{G})$) be the vector whose coordinates contain the values $\{d_0(v)\}_{v \in V(\overline{G})}$ (resp., $\{m(v)\}_{v \in V(\overline{G})}$), listed in non-increasing order.

Definition. Given two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{Z}^n$, we say that $\vec{x} \le \vec{y}$ if each coordinate of $\vec{x}$ has value that is at most the value in the corresponding coordinate of $\vec{y}$. Let X, Y be the vector of values of $\vec{x}$ and $\vec{y}$ listed in non-increasing order.

Lemma 2.5. Let $\vec{x}, \vec{y} \in \mathbb{Z}^n$. If $\vec{x} \le \vec{y}$, then $X \le Y$.

Proof. By definition, the first coordinate of X is $X_1 = \max\{x_i \mid 1 \le i \le n\}$, and so $X_1 = x_{i_1}$ for some $i_1 \in \{1, \ldots, n\}$. But by assumption, $X_1 = x_{i_1} \le y_{i_1} \le \max\{y_i \mid 1 \le i \le n\} = Y_1$.

More generally, for all $2 \le k \le n$, the k-th entry of X is less than or equal to at least k entries of $\vec{x}$, say $x_{i_1}, \ldots, x_{i_k}$. Further by assumption, for each h, $x_{i_h} \le y_{i_h}$. Since there are (at least) k coordinates of $\vec{y}$ that have value at least X_k, the value Y_k must be at least that large. Thus the k-th entry of X is less than or equal to at least k coordinates of $\vec{Y}$, and so $X_k \le Y_k$. Thus $X \le Y$. □

Corollary 2.6. Let $\overline{G}$ and $\widehat{G}$ be geometric realizations of a graph G, and suppose $\overline{G} \overset{f}{\preceq} \widehat{G}$. Then

1. $D_0(\overline{G}) \geq D_0(\widehat{G})$;
2. $M(\overline{G}) \leq M(\widehat{G})$.

In Figure 4, $D_0(\overline{K}_6) = (3, 3, 3, 2, 2, 1) \geq D_0(\widehat{K}_6) = (3, 2, 2, 2, 2, 1)$, while $M(\overline{K}_6) = (4, 4, 3, 3, 2, 2) \leq M(\widehat{K}_6) = (4, 4, 3, 3, 3, 2)$.

3 Posets for geometric paths

We now determine $\mathcal{P}_n$, the geometric homomorphism poset of the path P_n on n vertices, for $n = 2, \ldots, 6$, and we state some properties of this poset for general n. Throughout this section, we denote the vertices of P_n by $1, 2, \ldots, n$, and its edges by $e_i = \{i, i + 1\}, i = 1, \ldots, n - 1$. The following two lemmas are helpful in determining the geometric realizations of P_n.

Lemma 3.1. If a geometric graph $\overline{G}$ contains P_5 as a subgraph, with vertices and edges numbered as above and $e_1 \times e_3$ and $e_2 \times e_4$ are both crossings in $\overline{G}$, then so is $e_1 \times e_4$.

Proof. Suppose $\overline{G}$ has both of the crossings $e_1 \times e_3$ and $e_2 \times e_4$. Let ℓ be the line determined by edge e_1. Since e_1 crosses e_3, we may assume that vertex 3 lies above ℓ and vertex 4 lies below ℓ, as indicated in Figure 5. Let C be the cone with vertex 4 and sides extending through vertices 1 and 2 (indicated by dashed lines in Figure 5). For e_3 to cross e_1, both 3 and e_2 must be inside C. For e_4 to cross e_2, both 5 and e_4 must lie in the cone with vertex 4 and sides through 2 and 3, and 5 must also lie above e_2. This forces e_4 to cross e_1 in addition to crossing e_2. □
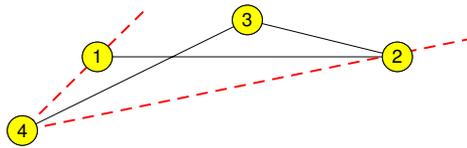

Figure 5: For use in the proof of Lemma 3.1

Lemma 3.2. A geometric realization of P_n has at most $(n - 2)(n - 3)/2$ edge crossings. Moreover, this bound is tight.

Proof. The only possible crossing edge pairs in any geometric realization of P_n are of the form $e_i \times e_j$ with $j - i \geq 2$; thus for any $i = 1, \ldots, n - 3$, there are $n - i - 2$ higher-numbered edges that can cross e_i. A straightforward algebraic calculation shows that $(n - 2)(n - 3)/2$ is an upper bound on the number of edge crossings. Figure 6 gives a geometric realization of P_7 that achieves this bound. □

To determine up to isomorphism all the possible geometric realizations of P_n, for $n = 2, \ldots, 5$, first list all sets whose elements are pairs of edges $e_i \times e_j$ with $j - i \geq 2$. Next
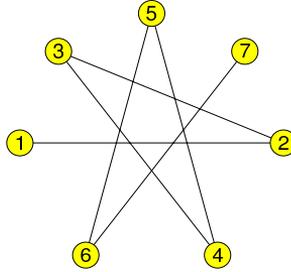
Figure 6: $\overline{P}_7$ with the maximum number of crossings

eliminate any sets that violate Lemma 3.1, and identify any sets that are equivalent under an automorphism of P_n. Recall that the only automorphisms of P_n are the identity and the map that reverses the order of the vertices. Finally, check that each of the remaining sets corresponds to a geometric realization.

To determine the structure of the geometric homomorphism poset, recall that by Proposition 2.2, $\overline{P}_n \preceq \widehat{P}_n$ if and only if there is $\tilde{f} \in \mathrm{Aut}(L(P_n))$ that induces a graph homomorphism from $EX(\overline{P}_n)$ to $EX(\widehat{P}_n)$. The graph $L(P_n) = P_{n-1}$ has only the two automorphisms mentioned above. Thus for each ordered pair of realizations, we need only check two automorphisms to see if they extend to color-preserving homomorphisms of the corresponding line/crossing graph.

For $n = 2, \ldots, 5$, all sets that satisfy Lemma 3.1 have geometric realizations. We state the poset results for $P_2, \ldots, P_5$ below.

Theorem 3.3. Let $\mathcal{P}_n$ be the poset of geometric realizations of P_n.

1. Each of $\mathcal{P}_2$ and $\mathcal{P}_3$ is trivial, containing only the plane realization.

2. $\mathcal{P}_4$ is a chain of two elements, in which the plane realization is the unique minimal element, and the realization with crossing $e_1 \times e_3$ is the unique maximal element.

3. $\mathcal{P}_5$ has the following five non-isomorphic geometric realizations and Hasse diagram as given in Figure 7: $0.1 = \emptyset$ (the plane realization) ; two 1-crossing realizations, $1.1 = \{e_1 \times e_3\} \equiv \{e_2 \times e_4\}$ and $1.2 = \{e_1 \times e_4\}$; a single 2-crossing realization, $2.1 = \{e_1 \times e_3, e_1 \times e_4\} \equiv \{e_1 \times e_4, e_2 \times e_4\}$; and a single 3-crossing realization, $3.1 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4\}$.

Proof. It is straightforward to find geometric realizations with the given sets of crossings. These realizations, together with their line/crossing graphs (which aid in determining the poset relations), appear in [4]. It follows from Lemmas 3.1 and 3.2 that there are no other realizations. □

For P_6, there is one set of crossing edge pairs that satisfies Lemma 3.1, but which does not correspond to a geometric realization of P_6, namely $\{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_5, e_3 \times e_5\}$. The following lemma shows that this set can also be eliminated.

Lemma 3.4. Suppose a geometric graph $\overline{G}$ contains P_6 as a subgraph, with vertices and edges numbered in the standard way. If $e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_5$, and $e_3 \times e_5$ are crossings in $\overline{G}$, then so is $e_2 \times e_4$.
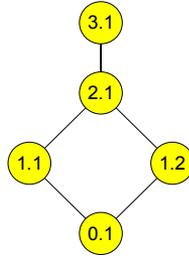
Figure 7: The Hasse diagram for $\mathcal{P}_5$

Proof. Since edges e_1 and e_3 cross, we may assume without loss of generality that e_2 is horizontal, and that e_1 and e_3 lie above e_2, as indicated in Figure 8. Since $\overline{G}$ contains the crossing $e_1 \times e_4$, vertex 5 is in one of the regions T, E, or S shown in Figure 8. If vertex 5 is in E or T, then e_5 cannot cross all three of the edges e_1, e_2, and e_3. Thus vertex 5 is in S, forcing the crossing $e_2 \times e_4$. □
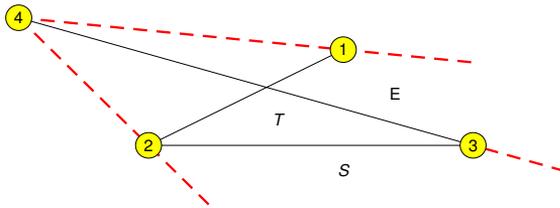
Figure 8: For use in the proof of Lemma 3.4

We are now able to list all the non-isomorphic geometric realizations of P_6 and give the Hasse diagram for $\mathcal{P}_6$.

Theorem 3.5. The poset $\mathcal{P}_6$ has the following thirty-one non-isomorphic geometric realizations and has Hasse diagram as given in Figure 9.

0 crossings: $0.1 = \emptyset$;

1 crossing: $1.1 = \{e_1 \times e_3\} \equiv \{e_3 \times e_5\}$, $1.2 = \{e_1 \times e_4\} \equiv \{e_2 \times e_5\}$, $1.3 = \{e_1 \times e_5\}$, $1.4 = \{e_2 \times e_4\}$;

2 crossings: $2.1 = \{e_1 \times e_3, e_1 \times e_4\}$, $2.2 = \{e_1 \times e_3, e_1 \times e_5\}$, $2.3 = \{e_1 \times e_3, e_2 \times e_5\}$, $2.4 = \{e_1 \times e_3, e_3 \times e_5\}$, $2.5 = \{e_1 \times e_4, e_1 \times e_5\}$, $2.6 = \{e_1 \times e_4, e_2 \times e_4\}$, $2.7 = \{e_1 \times e_4, e_2 \times e_5\}$, $2.8 = \{e_1 \times e_5, e_2 \times e_4\}$;

3 crossings: $3.1 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5\}$, $3.2 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4\}$, $3.3 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_5\}$, $3.4 = \{e_1 \times e_3, e_1 \times e_4, e_3 \times e_5\}$, $3.5 = \{e_1 \times e_3, e_1 \times e_5, e_2 \times e_5\}$, $3.6 = \{e_1 \times e_3, e_1 \times e_5, e_3 \times e_5\}$, $3.7 = \{e_1 \times e_4, e_1 \times e_5, e_2 \times e_4\}$ $3.8 = \{e_1 \times e_4, e_1 \times e_5, e_2 \times e_5\}$, $3.9 = \{e_1 \times e_4, e_2 \times e_4, e_2 \times e_5\}$;

4 crossings: $4.1 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4\} \equiv \{e_1 \times e_5, e_2 \times e_4, e_2 \times e_5, e_3 \times e_5\}$, $4.2 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_5\} \equiv \{e_1 \times e_4, e_1 \times e_5, e_2 \times e_5, e_3 \times e_5\}$, $4.3 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_3 \times e_5\} \equiv \{e_1 \times e_3, e_1 \times e_5, e_2 \times e_5, e_3 \times e_5\}$, 4.4

$= \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4, e_2 \times e_5\} \equiv \{e_1 \times e_4, e_2 \times e_4, e_2 \times e_5, e_3 \times e_5\}$, 4.5 = $\{e_1 \times e_3, e_1 \times e_4, e_2 \times e_5, e_3 \times e_5\}$, 4.6 = $\{e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5\}$;

    5 crossings: 5.1 = $\{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5\} \equiv \{e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5, e_3 \times e_5\}$, 5.2 = $\{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4, e_2 \times e_5, e_3 \times e_5\}$;

    6 crossings: 6.1 = $\{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5, e_3 \times e_5\}$.

Proof. It is straightforward to find geometric realizations with the given sets of crossings. These realizations, together with their line/crossing graphs (which aid in determining the poset relations), appear in [4]. It follows from Lemmas 3.1, 3.2, and 3.4 that there are no others. □

The following theorem lists some properties of $\mathcal{P}_n$ for $n \geq 3$.

Theorem 3.6. For $n \geq 3$, $\mathcal{P}_n$ has the following properties.

1. There is a unique minimal element, corresponding to the plane realization of P_n.

2. There is a unique maximal element, corresponding to the realization of P_n with $(n-2)(n-3)/2$ crossings.

3. $\mathcal{P}_n$ has a chain of size $(n-2)(n-3)/2 + 1$. In particular, for each c with $0 \leq c \leq (n-2)(n-3)/2$, there is at least one realization of P_n with exactly c crossings.

4. For $1 \leq k \leq n$, $\mathcal{P}_k$ is isomorphic to a sub-poset of $\mathcal{P}_n$.

Proof. Properties 1 and 2 are easily seen to be true. For Property 3, consider a geometric realization $\overline{P}_n$ of P_n with $c \geq 1$ crossings. Such a realization can be modified to create a new realization $\widehat{P}_n$ with $c - 1$ crossings, and with $\widehat{P}_n \prec \overline{P}_n$, by sliding the vertex n along edge e_{n-1} until it passes over a crossing edge, and then erasing the section of e_{n-1} that extends beyond this point. If e_{n-1} has no crossings, we slide vertices n and $n-1$ along edge e_{n-2}, erasing what remains of e_{n-1} and e_{n-2}, and so on. We can continue in a similar manner to remove one crossing at a time until there are none left; the process is illustrated in Figure 10. Since Lemma 3.2 guarantees that P_n has a realization with $(n-2)(n-3)/2$ crossings, Property 3 follows.

For Property 4, suppose we have some geometric realization of P_k. We can replace the uncrossed segment of edge e_{k-1} nearest to vertex k with a path from k to n to obtain a geometric realization of P_n with the same crossings. By doing this for each realization of P_k, we see that its poset of geometric realizations is isomorphic to a sub-poset of the poset of geometric realizations of P_n. □

A *cover* of an element x in a poset $\mathcal{P}$ is an element $y \in \mathcal{P}$ such that $x \prec y$ and no $z \in \mathcal{P}$ satisfies $x \prec z \prec y$. $\mathcal{P}$ is called a *graded* poset if there is a *rank* function $\rho : \mathcal{P} \to \mathbb{N}$ such that for all $x, y \in \mathcal{P}$, 1) all minimal elements have the same value under the rank function, 2) if $x \prec y$, then $\rho(x) < \rho(y)$, and 3) if y covers x, then $\rho(y) = \rho(x) + 1$. Note that if a poset is graded then all maximal chains between a given pair of elements must have the same length.

A reasonable conjecture for geometric homomorphism posets is that the number of edge crossings acts as a rank function. Condition 1 holds by Proposition 2.3. However, Condition 2 fails to hold in exactly one instance in $\mathcal{P}_6$: realization 6.1 covers realization 4.3, yet it has two more crossings. In fact, the poset $\mathcal{P}_6$ does not admit any rank function, because it has maximal chains between 0.1 and 6.1 which have different lengths: $0.1 \prec 1.4 \prec 2.8 \prec 3.7 \prec 4.6 \prec 5.1 \prec 6.1$ and $0.1 \prec 1.1 \prec 2.2 \prec 3.5 \prec 4.3 \prec 6.1$. Hence,

$\mathcal{P}_6$ is not a graded poset. It follows from Property 4 that $\mathcal{P}_n$ is not a graded poset for any $n \geq 6$.
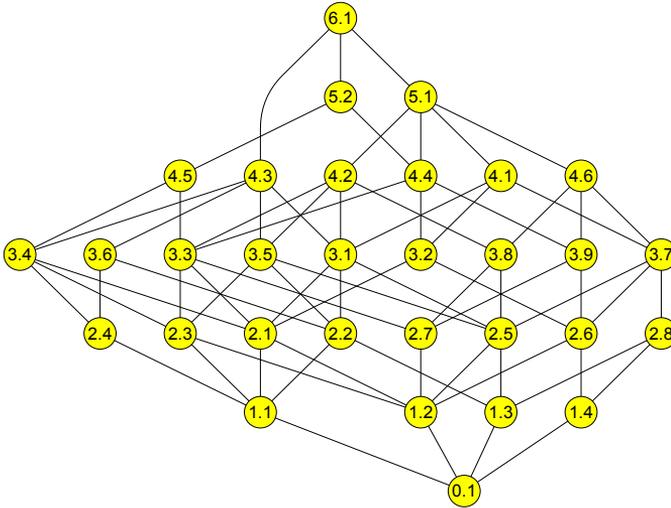

Figure 9: The Hasse diagram for $\mathcal{P}_6$

A *lattice* is a poset in which any two elements have a unique supremum (join) and unique infimum (meet). Figure 9 shows that $\mathcal{P}_6$ is not a lattice because (for example) realizations 3.5 and 3.1 have both 4.3 and 4.2 as suprema, and realizations 4.3 and 4.2 have both 3.5 and 3.1 as infima. It follows from Property 4 that $\mathcal{P}_n$ is not a lattice for any $n \geq 6$.
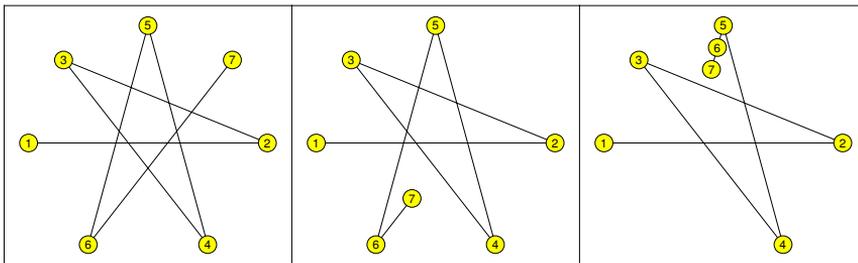
Figure 10: Geometric realizations of P_7 with 10 crossings, 6 crossings, and 3 crossings

4   Posets for geometric cycles

We now determine $\mathcal{C}_n$, the geometric homomorphism poset of the cycle C_n on n vertices, for $n = 3, \ldots, 6$, and we state some properties of this poset for general n. Throughout this section we denote the vertices of C_n by $1, 2, \ldots, n$, and its edges by $e_i = \{i, i+1\}, i = 1, \ldots, n-1$, and $e_n = \{n, 1\}$.

The maximum number of crossings in a geometric realization of C_n was determined in 1977 by Furry and Kleitman [6]; their results are summarized in the next lemma.

Lemma 4.1. [6] For $n \geq 3$, a geometric realization of C_n has at most $n(n-3)/2$ edge crossings if n is odd and $n(n-4)/2 + 1$ edge crossings if n is even. Moreover, these bounds are tight.

Figure 11 shows such a realization for $n = 10$.
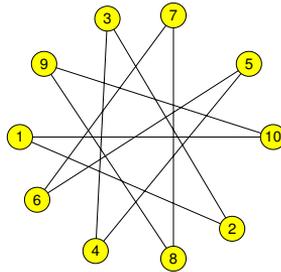
Figure 11: A realization of C_{10} with the maximum number of crossings

The techniques of the previous section can be used to find the elements of $\mathcal{C}_n$. For $3 \leq n \leq 5$, every set of crossing edge pairs that satisfies Lemma 3.1 corresponds to a geometric realization. For C_6, this is the case for all geometric realizations with at most two crossings. For realizations with three or more crossings, some cases require additional lemmas.

Lemma 4.2. Suppose $\overline{C}_n$ is a geometric realization of the cycle C_n that has crossings $e_i \times e_k$ and $e_j \times e_\ell$, where $i < j < k < \ell$. Then there is at least one additional crossing $e_\alpha \times e_\beta$ where $i \leq \alpha \leq k$ and $k \leq \beta \leq i \pmod{n}$ (and $\{\alpha, \beta\} \neq \{i, k\}$).

Proof. Suppose there is no such additional crossing. Place a vertex v at the crossing $e_i \times e_k$, subdividing each of those two edges. Starting at edge $\{v, i+1\}$ of the modified graph, follow the cycle in order of increasing vertex number, coloring each edge red, until the crossing $e_i \times e_k$ is reached again at edge $\{k, v\}$. Then follow the rest of the cycle, beginning at edge $\{v, k+1\}$, coloring each edge blue, until the final edge $\{i, v\}$ is reached. Note that edge e_j is red and edge e_ℓ is blue. The red cycle is a closed, but not necessarily simple, rectilinear curve in the plane. From the hypotheses of the lemma and our assumption that the conclusion is false, this red curve does not cross either of the edges $\{i, v\}$ and $\{v, k+1\}$, so these two edges lie in the same region of the plane determined by the red curve. If we now follow the blue curve starting at $k+1$, then the red-blue crossing $e_j \times e_\ell$ takes the blue curve into a different region of the plane determined by the red curve. But since we have assumed that the additional crossing of the lemma does not exist, the blue curve cannot return to end at vertex i, which is a contradiction. $\square$

Lemma 4.3 is a multi-part technical lemma. We prove the first part below; the proofs of the others, which are similar to the proof of Lemma 3.4, appear in [4].

Lemma 4.3. Let $\overline{C}_6$ be a geometric realization of the cycle C_6, with edges labeled consecutively, $e_1, e_2, \ldots, e_6$.

1. If $\overline{C}_6$ contains the crossings $e_1 \times e_3$, $e_1 \times e_4$, and $e_1 \times e_5$, then it doesn't contain the crossing $e_2 \times e_6$.

2. If $\overline{C}_6$ contains the crossings $e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4$, and $e_4 \times e_6$, then it also contains the crossing $e_2 \times e_5$.

3. If $\overline{C}_6$ contains the crossings $e_1 \times e_3, e_1 \times e_4, e_2 \times e_4$, and $e_2 \times e_5$, then it also contains at least one of the crossings $e_1 \times e_5, e_3 \times e_5, e_3 \times e_6$.

4. If $\overline{C}_6$ contains the crossings $e_1 \times e_3, e_1 \times e_4, e_2 \times e_5$, and $e_4 \times e_6$, then it also contains at least one of the crossings $e_2 \times e_4, e_3 \times e_5, e_3 \times e_6$.

5. If $\overline{C}_6$ contains the crossings $e_1 \times e_3, e_1 \times e_4, e_2 \times e_5$, and $e_3 \times e_6$, then it also contains at least one of the crossings $e_2 \times e_4, e_2 \times e_6, e_3 \times e_5, e_4 \times e_6$.

Proof. For part 1, suppose $\overline{C}_6$ contains the crossings $e_1 \times e_3, e_1 \times e_4$, and $e_1 \times e_5$. Let h be the line through edge e_1. Since e_1 crosses every edge of the path joining vertices 3 and 6, the vertices 3 and 6 lie on opposite sides of h. Thus the edges $e_2 = \{2, 3\}$ and $e_6 = \{6, 1\}$ also lie on opposite sides of h and so do not cross. The proofs of parts 2-5 are similar and appear in [4]. $\square$

Part 1 and its proof generalize to give us the following corollary.

Corollary 4.4. Let $\overline{C}_n$ be a geometric realization of the cycle C_n, where $n \geq 4$ is even. If $\overline{C}_n$ contains the crossings $e_1 \times e_3, e_1 \times e_4, \ldots, e_1 \times e_{n-1}$, then it doesn't contain the crossing $e_2 \times e_n$.

To determine the the elements of the poset $\mathcal{C}_6$, look at all possible sets of crossing edge pairs, and delete sets that don't satisfy Lemmas 3.1, 3.4, 4.2 or 4.3.

Next, identify those that are equivalent under an automorphism of C_n. There are $2n$ such automorphisms: each of the rotations and each of these composed with the reflection map. To determine the geometric homomorphisms among the remaining sets, recall that by Proposition 2.2, it suffices to look for automorphisms of the line graph $L(C_n)$ that extend to homomorphisms on the edge crossing graphs. Since $L(C_n) = C_n$, these automorphisms are precisely the $2n$ automorphisms mentioned above.

Theorem 4.5 lists the elements of the poset of geometric realizations of C_n for $3 \leq n \leq 6$; all nontrivial realizations are given up to isomorphism.

Theorem 4.5. Let $\mathcal{C}_n$ be the poset of geometric realizations of C_n.

1. $\mathcal{C}_3$ is trivial, containing only the plane realization.

2. $\mathcal{C}_4$ is a chain of two elements, in which the plane realization is the unique minimal element, and the realization with crossing $e_1 \times e_3$ is the unique maximal element.

3. $\mathcal{C}_5$ is a chain of five elements: the plane realization $0.1 = \emptyset$, $1.1 = \{e_1 \times e_3\}$, $2.1 = \{e_1 \times e_3, e_1 \times e_4\}$, $3.1 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4\}$, and $5.1 = \{e_1 \times e_4, e_2 \times e_4, e_2 \times e_5, e_3 \times e_4\}$.

4. $\mathcal{C}_6$ has the following twenty-six non-isomorphic geometric realizations and has Hasse diagram as given in Figure 12:

   0 crossings: $0.1 = \emptyset$ (the plane realization);

   1 crossing: $1.1 = \{e_1 \times e_3\}, 1.2 = \{e_1 \times e_4\}$;

   2 crossings: $2.1 = \{e_1 \times e_3, e_1 \times e_4\}, 2.2 = \{e_1 \times e_3, e_1 \times e_5\}, 2.3 = \{e_1 \times e_3, e_4 \times e_6\}$;

3 crossings: $3.1 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5\}, 3.2 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4\}, 3.3 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_5\}, 3.4 = \{e_1 \times e_3, e_1 \times e_4, e_3 \times e_5\}, 3.5 = \{e_1 \times e_4, e_3 \times e_6\}, 3.6 = \{e_1 \times e_3, e_1 \times e_4, e_4 \times e_6\}, 3.7 = \{e_1 \times e_3, e_1 \times e_5, e_3 \times e_5\}, 3.8 = \{e_1 \times e_4, e_2 \times e_5, e_3 \times e_6\};$

4 crossings: $4.1 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4\}, 4.2 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_5\}, 4.3 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_3 \times e_5\}, 4.4 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_5, e_3 \times e_5\}, 4.5 = \{e_1 \times e_3, e_1 \times e_4, e_3 \times e_6, e_4 \times e_6\};$

5 crossings: $5.1 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5\}, 5.2 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4, e_2 \times e_5, e_3 \times e_5\}, 5.3 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_4, e_2 \times e_5, e_3 \times e_6\}, 5.4 = \{e_1 \times e_3, e_1 \times e_4, e_2 \times e_5, e_3 \times e_6, e_4 \times e_6\};$

6 crossings: $6.1 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5, e_3 \times e_5\}, 6.2 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5, e_3 \times e_6\};$

7 crossings: $7.1 = \{e_1 \times e_3, e_1 \times e_4, e_1 \times e_5, e_2 \times e_4, e_2 \times e_5, e_3 \times e_6, e_4 \times e_6\}.$

Proof. It is straightforward to find geometric realizations with the given sets of crossings. These realizations, together with their line/crossing graphs (which aid in determining the poset relations) appear in [4]. It follows from Lemmas 3.1, 3.4, 4.1, 4.2 and 4.3 that there are no other realizations. □
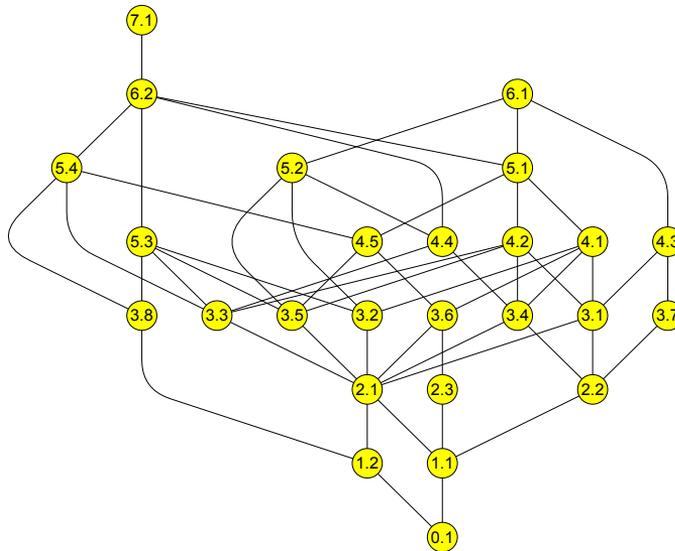

Figure 12: The Hasse diagram for $\mathcal{C}_6$

Unlike the case of $\mathcal{P}_n$, we see from the geometric realizations of C_5 that not every possible number of crossings up to the maximum is necessarily achieved. On the other hand, C_6 has at least one realization with each number of crossings from 0 up to its maximum of 7. Furry and Kleitman have shown that this is representative of the geometric realizations of all odd and even cycles. This is stated in Theorem 4.6 below.

Theorem 4.6. [6] For n even, $n \geq 4$, $\overline{C}_n$ can have any number of crossings from 0 up to the maximum of $n(n-4)/2 + 1$. For n odd, $n \geq 3$, $\overline{C}_n$ can have any number of crossings from 0 up to the maximum of $n(n-3)/2$, except there is no geometric realization with $n(n-3)/2 - 1$ crossings.

Finally we mention three more properties that are true in general for any geometric realization of the cycle C_n. The first two are obvious, and the third is easy to see, since any realization of C_n can be replaced by one of C_{n+1}, by subdividing the edge e_n into two edges, e_n and e_{n+1}, so that the new edge e_{n+1} has no crossings.

Theorem 4.7. For $n \geq 3$, the poset $\mathcal{C}_n$ has the following properties.

1. There is a unique minimal element, corresponding to the plane realization of C_n.

2. There is a unique maximal element, corresponding to the geometric realization with the maximum number of crossings, as given in Theorem 4.6.

3. For $3 \leq k \leq n$, the poset $\mathcal{C}_k$ is isomorphic to a sub-poset of $\mathcal{C}_n$.

Note that $\mathcal{C}_6$ is not a graded poset because it has maximal chains between 0.1 and 6.1 which have different lengths: $0.1 \prec 1.1 \prec 2.2 \prec 3.7 \prec 4.3 \prec 6.1$ and $0.1 \prec 1.1 \prec 2.2 \prec 3.1 \prec 4.1 \prec 5.1 \prec 6.1$. Thus by Property 2, for all $n \geq 6$, $\mathcal{C}_n$ is not a graded poset. Also, $\mathcal{C}_6$ is not a lattice because, for example, realizations 2.1 and 2.2 do not have a unique supremum. By Property 2, $\mathcal{C}_n$ is not a lattice for all $n \geq 6$.

5 Posets for geometric cliques

We now determine $\mathcal{K}_n$, the geometric homomorphism poset of the clique K_n for $n = 3, \ldots, 6$, and we state some properties of this poset for general n. Throughout this section we denote the vertices of K_n by $1, 2, \ldots, n$, and its edges by $e_{ij} = \{i, j\}, i \neq j \in \{1, \ldots, n\}$.

In [8], Harborth and Thürmann give all non-isomorphic geometric realizations of K_n for $3 \leq n \leq 6$. Recall that their definition for geometric isomorphism is stricter than the definition being used here. However, that only means that in general, our set of non-isomorphic geometric realizations may be smaller than theirs. That is, two geometric realizations that Harborth and Thürmann consider non-isomorphic, we may consider isomorphic. However, in the cases K_3, K_4, K_5, K_6, all pairs that are non-isomorphic according to Harborth are also non-isomorphic according to us.

Theorem 5.1. Let $\mathcal{K}_n$ be the poset of geometric realizations of K_n.

1. $\mathcal{K}_3$ is trivial, containing only the plane realization.

2. $\mathcal{K}_4$ is a chain of two elements, in which the plane realization is the unique minimal element, and the realization with crossing $e_{1,3} \times e_{2,4}$ is the unique maximal element.

3. $\mathcal{K}_5$ is a chain of three elements: $1.1 = \{e_{3,5} \times e_{2,4}\}$, $3.1 = \{e_{1,4} \times e_{2,5}, e_{1,4} \times e_{3,5}, e_{2,4} \times e_{3,5}, \}$, and $5.1 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,5}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{3,5}, e_{2,4} \times e_{3,5}\}$.

4. $\mathcal{K}_6$ has Hasse diagram as given in Figure 13 with fifteen non-isomorphic geometric realizations:

   3 crossings: $3.1 = \{e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{3,5} \times e_{4,6}\}$;

   4 crossings: $4.1 = \{e_{1,3} \times e_{2,6}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{5,6}, e_{3,5} \times e_{4,6}\}$;

5 crossings: $5.1 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,6}, e_{2,4} \times e_{3,6}\}$, $5.2 = \{e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{3,6}, e_{3,5} \times e_{4,6}\}$;

6 crossings: $6.1 = \{e_{1,4} \times e_{2,5}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,6}, e_{2,4} \times e_{3,6}, e_{2,5} \times e_{3,6}, e_{2,5} \times e_{4,6}\}$;

7 crossings: $7.1 = \{e_{1,3} \times e_{2,5}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{5,6}, e_{1,6} \times e_{2,5}, e_{3,5} \times e_{4,6}\}$, $7.2 = \{e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{3,5}, e_{1,5} \times e_{4,6}, e_{2,4} \times e_{3,5}, e_{2,5} \times e_{4,6}, e_{3,5} \times e_{4,6}\}$;

8 crossings: $8.1 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{5,6}, e_{1,6} \times e_{2,4}, e_{2,4} \times e_{3,5}, e_{2,4} \times e_{5,6}, e_{3,5} \times e_{4,6}\}$, $8.2 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,6}, e_{1,5} \times e_{2,6}, e_{1,5} \times e_{3,6}, e_{1,5} \times e_{4,6}, e_{2,4} \times e_{3,6}\}$;

9 crossings: $9.1 = \{e_{1,3} \times e_{2,5}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{3,6}, e_{2,5} \times e_{3,6}, e_{2,5} \times e_{4,6}, e_{3,5} \times e_{4,6}\}$, $9.2 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,5}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,6}, e_{2,4} \times e_{3,6}, e_{2,5} \times e_{3,6}, e_{2,5} \times e_{4,6}\}$;

10 crossings: $10.1 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,5}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{5,6}, e_{1,6} \times e_{2,5}, e_{2,4} \times e_{3,5}, e_{2,4} \times e_{3,6}, e_{3,5} \times e_{4,6}\}$;

11 crossings: $11.1 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,5}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{5,6}, e_{1,6} \times e_{2,4}, e_{1,6} \times e_{2,5}, e_{2,4} \times e_{3,5}, e_{2,4} \times e_{5,6}, e_{3,5} \times e_{4,6}\}$;

12 crossings: $12.1 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,5}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{3,6}, e_{2,4} \times e_{3,5}, e_{2,4} \times e_{3,6}, e_{2,5} \times e_{3,6}, e_{2,5} \times e_{4,6}, e_{3,5} \times e_{4,6}\}$;

15 crossings: $15.1 = \{e_{1,3} \times e_{2,4}, e_{1,3} \times e_{2,5}, e_{1,3} \times e_{2,6}, e_{1,4} \times e_{2,5}, e_{1,4} \times e_{2,6}, e_{1,4} \times e_{3,5}, e_{1,4} \times e_{3,6}, e_{1,5} \times e_{2,6}, e_{1,5} \times e_{3,6}, e_{1,5} \times e_{4,6}, e_{2,4} \times e_{3,5}, e_{2,4} \times e_{3,6}, e_{2,5} \times e_{3,6}, e_{2,5} \times e_{4,6}, e_{3,5} \times e_{4,6}\}$.

Proof. Since $\text{Aut}(K_n)$ contains all possible permutations of the vertices, it does not make our job easier to first restrict our search for homomorphisms to those that are induced by automorphisms of the underlying abstract graph. Thus we use the tools of Subsection 2.2 rather than those of Subsection 2.1. Drawings of the realizations of K_5 and K_6, as well justifications of the poset relations and non-relations, appear in [4]. ☐
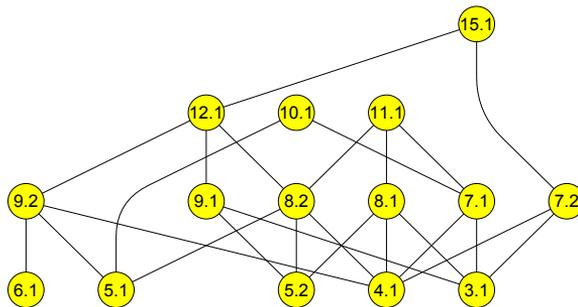

Figure 13: The Hasse diagram for $\mathcal{K}_6$

Observe that $\mathcal{K}_6$ has five minimal elements and three maximal ones. As with cycles, not every possible number of crossings, from 3 up to the maximum of 15, is achieved; there are no realizations of K_6 containing 13 crossings or 14 crossings. Clearly, the number of

edge crossings cannot act as a rank function. In fact, $\mathcal{K}_6$ is not a graded poset because it has maximal chains between 3.1 and 15.1 of different lengths: $3.1 \prec 7.2 \prec 15.1$ and $3.1 \prec 9.1 \prec 12.1 \prec 15.1$. Moreover, $\mathcal{K}_6$ is not a lattice, because realizations 3.1 and 4.1 do not have a unique supremum.

Although $\mathcal{K}_6$ has no rank function, the function taking a realization to the number of vertices in the boundary of its convex hull is order-preserving. In Figure 13, all realizations displayed on the bottom level of the Hasse diagram have 3 vertices in the boundary of the convex hull, those on the second level have 4, those on the third level have 5 and realization 15.1 has 6.

Theorem 5.2. For all $n \geq 3$, $\mathcal{K}_n$ contains a maximal chain of length $n-2$. More precisely, $\mathcal{K}_n$ contains a chain of the form

$$\overline{H}_3 \prec \overline{H}_4 \prec \cdots \prec \overline{H}_n$$

where $\overline{H}_k$ denotes a geometric realization of K_n with k vertices on the boundary of its convex hull.

Proof. We start with a template; consider the circle $x^2 + (y+1)^2 = 4$ in the xy plane, together with the two tangent lines at $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$ that intersect at $(0, 3)$. Place $n-1$ vertices along the upper portion of the circle, starting at $(-\sqrt{3}, 0)$ and ending at $(\sqrt{3}, 0)$; they should be roughly evenly spaced, but in general position. Label the leftmost one n, and the remaining ones $2, 3, \ldots, n-1$ from left to right. Add another vertex at $(0, 3)$ and label it 1. To complete the template, for each $k \in \{2, 3, \ldots, n-2\}$, add a ray from vertex 1 through vertex k and mark where it intersects the lower portion of the circle with $(n-k+1)^*$. See Figure 14.

Joining all pairs of vertices in the template with an edge gives us $\overline{H}_3$. Note that the boundary of its convex hull consists of vertices $1, n$ and $n-1$ and that all crossings in $\overline{H}_3$ occur in the geometric subgraph induced by the vertices $2, 3, \ldots, n$. To get $\overline{H}_4$, slide vertex $n-1$ clockwise along the circle to position $(n-1)^*$. Then slide vertices 2 through $n-2$ clockwise 'one spot' along the upper portion of the circle. This is now a geometric realization of K_n in which the boundary of the convex hull consists of the four vertices $1, n-2, n-1$ and n. Since vertices $2, 3, \ldots, n$ are still in convex position, no crossings of $\overline{H}_3$ have been lost. However the edge $\{1, n-1\}$ now crosses edges $\{2, n\}, \{3, n\}, \ldots, \{n-2, n\}$. Thus $\overline{H}_3 \prec \overline{H}_4$. To get $\overline{H}_5$, move vertex $n-2$ to position $(n-2)^*$ and shift vertices 2 through $n-3$ clockwise another 'one spot' along the circle. This gives a geometric realization in which the boundary of the convex hull consists of the vertices $1, n-3, n-2, n-1$ and n. Again, no edge crossings have been lost, but edge $\{1, n-2\}$ now crosses $\{2, n\}, \{3, n\}, \ldots, \{n-3, n\}$. Iterating this process yields the chain described in the theorem.

To prove the maximality of this chain, note that its final element $\overline{H}_n$ has all n vertices in convex position and so has the maximum number of crossings of any realization of K_n and therefore has no successor in $\mathcal{K}_n$. Next, suppose that $f : \overline{H} \to \overline{H}_3$ is an injective geometric homomorphism. By Proposition 2.4, all edges incident to vertex $v = f^{-1}(1)$ must be uncrossed. Let w, x, y, z be any other four vertices in $\overline{H}$; if they are not in convex position, then one of them, say w, must lie in the interior of the convex hull of the other three. This would imply that the edge $\{v, w\}$ is crossed in $\overline{H}$, a contradiction. Hence all $n-1$ other vertices in $\overline{H}$ lie in convex position, implying that in fact $\overline{H} \cong \overline{H}_3$. $\qquad\square$
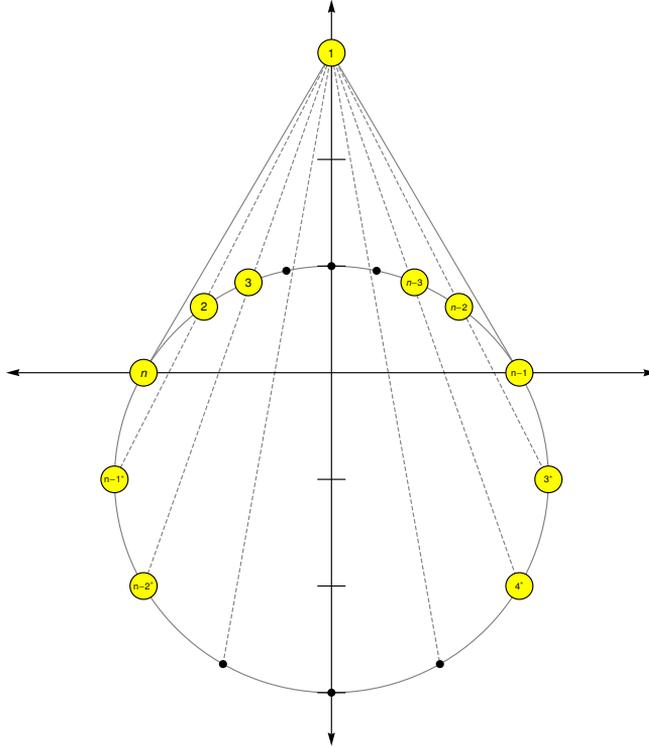
Figure 14: Template for the proof of Theorem 5.2

Each of the posets $\mathcal{K}_4$ and $\mathcal{K}_5$ is precisely the chain given in Theorem 5.2. Within $\mathcal{K}_6$, the chain constructed in Theorem 5.2 is $5.1 \prec 9.2 \prec 12.1 \prec 15.1$.

6 Open questions

1. Are there (closed or recursive) formulas for the number of elements in $\mathcal{P}_n$ or $\mathcal{C}_n$?

2. For $3 \le k \le n$, is $\mathcal{K}_k$ a sub-poset of $\mathcal{K}_n$?

3. If $\overline{K}_n \prec \widehat{K}_n$, must the number of vertices in the convex hull of $\overline{K}_n$ be strictly less than that of $\widehat{K}_n$? If so, then the chain constructed in Theorem 5.2 is a maxi*mum* chain.

4. We saw that $\mathcal{K}_6$ has a maximal chain of length 2. What is the length of a smallest possible maximal chain in $\mathcal{K}_n$?

5. What is the geometric homomorphism poset for other common families of graphs? In [5], Cockburn and Song have determined the geometric homomorphism poset $\mathcal{K}_{2,n}$ for one family of complete bipartite graphs.

References

[1] M. O. Albertson and D. L. Boutin, Distinguishing geometric graphs, *J. Graph Theory* **53** (2006), 135–150.

[2] M. O. Albertson and D. L. Boutin, Automorphisms and distinguishing numbers of geometric cliques, *Discrete Comput. Geom.* **39** (2008), 778–785.

[3] D. Boutin and S. Cockburn, Geometric graph homomorphisms, *J. Graph Theory* **69** (2012), 97–113.

[4] D. Boutin, S. Cockburn, A. Dean, and A. Margea, Posets of geometric graphs, with appendices, *arXiv.org*, 2011.

[5] S. Cockburn and Y. Song, The homomorphism poset of $K_{2,n}$, *arXiv.org*, 2012.

[6] W. Furry and D. Kleitman, Maximal rectilinear crossing of cycles, *Studies in Appl. Math.*, **56** (1977), 159–167.

[7] H. Harborth, Drawings of the cycle graph, *Congr. Numer.* **66** (1988), 15–22, Nineteenth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Baton Rouge, LA, 1988).

[8] H. Harborth and Ch. Thürmann, Numbers of edges without crossings in rectilinear drawings of the complete graph, *Congr. Numer.* **119** (1996), 79–83.

[9] P. Hell and J. Nešetřil, *Graphs and homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2004.

[10] J. Nešetřil, Homomorphisms of derivative graphs, *Discrete Math.* **1** (1971/72), 257–268.

[11] H. Whitney, Congruent Graphs and the Connectivity of Graphs, *Amer. J. Math.*, **54** (1932), 150–168.